RISK, THE PRICING OF CAPITAL ASSETS, AND THE EVALUATION OF INVESTMENT PORTFOLIOS*

MICHAEL C. JENSEN†

I. INTRODUCTION

A. RISK AND THE EVALUATION OF PORTFOLIOS

The main purpose of this study is the development of a model for evaluating the performance of portfolios of risky assets. In evaluating the performance of portfolios the effects of differential risk must be taken into consideration.[1] If investors are generally averse to risk, they will prefer (*ceteris paribus*) more certain income streams to less certain streams. Under these conditions investors will accept additional risk only if they are compensated for it in the form of higher expected future returns. Thus, in a world dominated by risk-averse investors, a risky portfolio must be expected to yield higher returns than a less risky portfolio, or it would not be held.

The portfolio evaluation model developed below incorporates these risk aspects explicitly by utilizing and extending recent theoretical results by Sharpe [52] and Lintner [37] on the pricing of capital assets under uncertainty. Given these results, a measure of portfolio "performance" (which measures only a manager's ability to forecast security prices) is defined as the difference between the actual returns on a portfolio in any particular holding period and the expected returns on that portfolio conditional on the riskless rate, its level of "systematic risk," and the actual returns on the market portfolio. Criteria for judging a portfolio's performance to be *neutral*, *superior*, or *inferior* are established.

A measure of a portfolio's "efficiency" is also derived, and the criteria for judging a portfolio to be *efficient*, *superefficient*, or *inefficient* are defined. It is also shown that it is strictly impossible to define a measure of efficiency solely in terms of ex post observable variables. In addition, it is shown that there exists a natural relationship between the measure of portfolio performance and the measure of efficiency.

* The research on this study was supported by fellowship grants from the U.S. Steel Foundation, the American Banking Association and a research grant from the Research Fund in Finance made available by the University of Chicago Graduate School of Business. Extensive computer time at the 7094 Computation Center at the University of Chicago was financed by the Graduate School of Business, and the College of Business of the University of Rochester provided additional time at the 360 Computation Center at the University of Rochester.

† Assistant professor, College of Business Administration, University of Rochester. I wish to acknowledge a great debt to my dissertation committee; Eugene Fama (chairman), Lawrence Fisher, Merton Miller (who originally suggested this area of research to me), and Harry Roberts, all of whom have given generously of their time and ideas and have continually forced me to rethink and defend my position on numerous issues. I am especially indebted to Professor Fama for his penetrating criticisms of several drafts of this paper. I also wish to thank the members of the Finance Workshop at the University of Chicago for many stimulating and helpful discussions, especially M. Blume, P. Brown, D. Duvel, and M. Scholes. I have also benefited from conversations with M. Geisel, F. Black, and Professors Peter Pashigian, Arnold Zellner, Donald Gordon, and Julian Keilson.

[1] Risk, a critical concept in this paper, will be defined and discussed extensively in Sections II, III, and V.

B. SECURITY PRICE MOVEMENTS, EFFICIENT MARKETS, MARTINGALES AND THEIR IMPLICATION FOR SECURITY ANALYSIS

There has recently been considerable interest in the behavior of security prices in an "efficient market" and more specifically in the martingale hypothesis of price behavior. There seem to be two different forms of the hypothesis which have arisen out of differing definitions of the concept of an "efficient market," definitions which are seldom explicitly enumerated.

One can define a weakly efficient market in the following sense: Consider the arrival in the market of a new piece of information concerning the value of a security. A weakly efficient market is a market in which it may take time to evaluate this information with regard to its implications for the value of the security. Once this evaluation is complete, however, the price of the security immediately adjusts (in an unbiased fashion) to the new value implied by the information. In such a weakly efficient market, the past price series of a security will contain no information not already impounded in the current price. Mandelbrot [39] and Samuelson [47] have rigorously demonstrated that prices in such a market will follow a submartingale—that is, the expected value of all future prices $X(t+\tau)$, $(\tau = 1, \ldots \infty)$, as of time t is independent of the *sequence* of past prices $X(t-\tau)$, $(\tau = 1, \ldots \infty)$, and is equal to:

$$E[X(t+\tau)\,|\,X(t),X(t-1),X(t-2),\,\ldots] = E[X(t+\tau)\,|\,X(t)] \qquad (1.1)$$
$$= f(\tau)X(t)\ ,$$

where $f(\tau)$ is the "normal" accumulation rate.

Thus, in a market in which security prices behave as a submartingale of the form of (1.1), forecasting techniques[2] which use only the sequence of past prices to forecast future prices are doomed to failure. The best forecast of future price is merely the present price plus the normal expected return over the period.

The stock market has been subjected to a great deal of empirical investigation aimed at determining whether (1.1) is an adequate description of the serial behavior of stock prices.[3] The available evidence suggests that it is highly unlikely that an investor or portfolio manager will be able to use the past history of stock prices alone (and hence mechanical trading rules based on these prices[4]) to increase his profits.

However, the conclusion that stock prices follow a submartingale of the form (1.1) does not imply that an investor cannot increase his profits by improving his ability to predict and evaluate the consequences of *future* events affecting stock prices. Indeed, it has been suggested by Fama [12] that the existence of sophisticated "market participants" who are adept at evaluating current information and predicting future events is one of the reasons why market prices at any point in time represent an unbiased estimate of "true" values and adjust rapidly, and accurately, to new information regarding these values.[5]

This brings us to an alternative definition of an "efficient market," that is, one in which *all* past information available up to time t is impounded in the current

[2] Charting techniques are one example.

[3] See especially the work by Fama [12] and the works reprinted in Cootner [9].

[4] For an example of the testing of one class of such rules see [12].

[5] For an example of an examination of such adjustment, see Fama *et al.* [19].

price. Within this definition of an efficient market the Mandelbrot-Samuelson proofs imply that the martingale property can be written as

$$E(X(t+\tau)\,|\,\theta_t) = f(\tau)X(t)\ , \quad (1.2)$$

where the conditioning variable θ_t now represents all information available at time t.[6] The reader will note that (1.2) is a much stronger form of the martingale hypothesis than (1.1), which is conditioned only on the past price series. As such (1.2) might be labeled the "strong" form of the martingale hypothesis and (1.1) the "weak" form.[7] Indeed, if security prices follow a martingale of the strong form, no analyst will be able to earn above-average returns by attempting to predict future prices on the basis of *past* information. The only individual able to earn superior returns will be that person who occasionally is the first to acquire a new piece of information not generally available to others in the market. But as Roll [46] argues, in attempting to act immediately on this information, this individual (or group of individuals) will insure that the effects of this new information are quickly impounded in the security's price. Furthermore, if new information of this type arises randomly, no individual will be able to assure himself of systematic receipt of such information. Therefore, while an individual may occasionally realize such windfall returns, he will be unable to earn them systematically through time.

While the weak form of the martingale hypothesis is well substantiated by empirical evidence, the strong form of the hypothesis has not as yet been subjected to extensive empirical tests.[8] The model developed below will allow us to submit the strong form of the hypothesis to such an empirical test—at least to the extent that its implications are manifested in the success or failure of one particular class of extremely well-endowed security analysts.

C. APPLICATIONS OF THE MODEL

The portfolio evaluation model developed below will be used to examine the results achieved by the portfolio managers of open end mutual funds in an attempt to answer the following questions:

1) Do the historical patterns of risk and return observed for our sample of portfolios of risky assets indicate a predominance of risk aversion in the capital markets? If so, do these patterns confirm the implications of the theoretical models of capital asset pricing founded on the assumption of risk aversion?

2) Have open-end mutual funds in general exhibited an ability to select portfolios which earn returns higher than those they may have been expected to earn given their level of risk? Alternatively, have they exhibited an ability to earn returns higher than those which could have been earned by a naïve selection policy consistent with the theory of capital asset pricing?

The main conclusions will be:

1) The observed historical patterns of systematic risk and return for the mutual funds in the sample are consistent with

[6] See Roll [46] for a discussion of the reasoning which leads to (1.2).

[7] To the best of my knowledge, this terminology is due to Harry Roberts, who used it in an unpublished speech entitled "Clinical vs. Statistical Forecasts of Security Prices," given at the Seminar on the Analysis of Security Prices sponsored by the Center for Research in Security Prices at the University of Chicago, May, 1967.

Subsequent to writing the present paper, an article by Shelton [59] has appeared which contains a very similar statement of the hypotheses.

[8] The only evidence on this question that I am aware of is contained in Fama *et al.* [19], and that evidence suggests that security prices adjust rapidly and in an unbiased fashion to new information.

the joint hypothesis that the capital asset pricing model is valid and that the mutual fund managers on the average are unable to forecast future security prices.

2) If we assume that the capital asset pricing model is valid, then the empirical estimates of fund performance indicate that the fund portfolios were "inferior" after deduction of all management expenses and brokerage commissions generated in trading activity. In addition, when all management expenses and brokerage commissions are added back to the fund returns and the average cash balances of the funds are assumed to earn the riskless rate, the fund portfolios appeared to be just "neutral." Thus, it appears that on the average the resources spent by the funds in attempting to forecast security prices do not yield higher portfolio returns than those which could have been earned by equivalent risk portfolios selected (*a*) by random selection policies or (*b*) by combined investments in a "market portfolio" and government bonds.

3) Based on the evidence summarized above, we conclude that as far as these 115 mutual funds are concerned, prices of securities seem to behave according to the "strong" form of the martingale hypothesis. That is, it appears that the current prices of securities completely capture the effects of *all* information available to these 115 mutual funds. Therefore, their attempts to analyze past information more thoroughly have not resulted in increased returns.

Although these results certainly do not imply that the strong form of the martingale hypothesis holds for all investors and for all time, they provide strong evidence in support of that hypothesis. One must realize that these analysts are extremely well endowed.[9] Moreover, they operate in the securities markets every day and have wide-ranging contacts and associations in both the business and the financial communities. Thus, the fact that they are apparently unable to forecast returns accurately enough to recover their research and transactions costs is a striking piece of evidence in favor of the strong form of the martingale hypothesis—at least as far as the extensive subset of information available to these analysts is concerned.

4) The evidence also indicates that, while the portfolios of the funds on the average are "inferior" and "inefficient," this is due mainly to the generation of too many expenses. Since the evidence indicates that the portfolios on the average are very well diversified, they are "inefficient" mainly because of the generation of too many expenses.

D. AN OUTLINE OF THE STUDY

The portfolio evaluation model is developed in Sections II–V. The foundations of the model are discussed in Section II, which proceeds with a brief review of: (1) a theory of rational choice under uncertainty; (2) the normative theory of portfolio selection; and finally (3) a closely associated theoretical model of the pricing of capital assets under uncertainty.

Section III contains a development of the evaluation model under the assumption of homogeneous investor horizon periods. The "market model" and the concept of "systematic risk" are defined, and their application to the evaluation problem is discussed in detail. Finally, measures of portfolio "performance" are derived under alternative assumptions regarding the existence of finite or infinite variances for the distributions of returns.

[9] For example, the total income received by eighty-six investment advisory firms from open-end investment companies amounted to $32.6 million in the fiscal years ending in 1960–61 (cf. Friend *et al.* [26, p. 497]).

Section IV contains a discussion of the "horizon problem," a solution to it, and the extension of the evaluation model to a world in which investors have heterogeneous horizon periods.

Section V contains a discussion of the evaluation criteria, the derivation of a measure of "efficiency," and an examination of the relationship between the concepts of "performance" (defined in Section III) and "efficiency."

Section VI presents a discussion of (1) the empirical estimates of the concept of "systematic risk" for 115 mutual funds, (2) some empirical tests of the assumptions of the "market model," and (3) an application of the model to the evaluation of these 115 mutual fund portfolios.

Section VII contains a summary of the theoretical and empirical results and their implications and a brief discussion of some of the main criticisms which will undoubtedly arise regarding the findings.

The reader interested mainly in the empirical applications of the model discussed in Section VI-B may obtain the general flavor of the model by a close examination of Sections III-A and III-B, a cursory examination of Section IV, and a close examination of Section V, which presents a number of crucial points.

II. THE FOUNDATIONS OF THE MODEL

A. A THEORY OF RATIONAL CHOICE UNDER UNCERTAINTY

The expected utility maxim.—The problem of choice under uncertainty is characterized by situations in which an individual faces a set of alternative actions, and the outcomes associated with these actions are subject to probability distributions. We shall assume in the development to follow that a rational individual, when faced with a choice under conditions of uncertainty, acts in a manner consistent with the expected utility maxim. That is, he acts as if he (1) attaches numbers (utilities) to each possible outcome and (2) chooses that option (or strategy) with the largest expected value of utility.[10]

The consumption-investment problem.—Accepting the expected utility maxim as the objective function, the general problem of the investor in an uncertain world can be stated as the maximization of the expected value of

$$U = U(C_1, C_2, \ldots, C_t, \ldots, C_T, W_T) \,, \quad (2.1)$$

where C_t is the real value of consumption in period t, T is the time of death (which, of course, is a random variable), W_T is the bequest, and U is the utility of the investor's lifetime consumption pattern. The portfolio problem arises within this framework when the investor has assets in one period which he does not wish to consume in that period, but rather desires to carry over into the next period. His portfolio problem at any time t then becomes the selection of a combination of investments which yield him maximum expected utility.

While the consumption-investment problem is most certainly a multiperiod problem, the lack of a well-developed multiperiod theory of choice under uncertainty has led most researchers to assume that the portfolio decision can be treated as a *single-period* decision to be made independently of the consumption decision.[11] Necessary and sufficient con-

[10] An axiomatic derivation of the expected utility maxim is given by Von Neuman and Morgenstern [65] and Markowitz [42]. In chapters x–xiii, Markowitz [42] gives a thorough exposition of the hypothesis and its implications for the portfolio decision in particular.

[11] See, for example, references 7, 13, 14, 22, 36, 37, 40, 42, 43, 51, 52, 61, and 62, all of which (either implicitly or explicitly) are single-period utility of terminal wealth models. That is, they assume the investor's problem can be characterized by the maximization of the expected value of $U(W_{t+1})$, where W_{t+1} is the terminal wealth of the portfolio one period hence.

ditions under which these simplifying assumptions will lead to an optimal solution of the unrestricted multiperiod problem of (2.1) have been determined only for a very restricted class of utility functions (cf. Hakansson [29] and Mossin [44]. However, Fama [15] has shown under very general conditions that, while the investor must solve a T-period problem like (2.1) in order to make consumption and investment decisions for period 1, he will behave *as though* he were a single-period expected utility maximizer. That is, the investor will appear to behave *as though* he were maximizing

$$E[U(C_t, W_{t+1})] , \qquad (2.2)$$

where C_t and W_{t+1} are, respectively, the value of consumption in period t and the terminal value of the portfolio at the end of period t, and his decision variables are C_t and x_i, the fraction of the portfolio invested in the ith asset. In addition, if all assets are perfectly liquid[12] and infinitely divisible and there are no taxes,[13] Fama [16] has also demonstrated that, in solving the simultaneous consumption-investment problem of (2.2), the investor will always choose a portfolio which is efficient in terms of single-period parameters. That is, the investor will always choose a portfolio which is efficient in the sense that for the period under consideration it provides maximum expected return for given level of risk and minimum risk for given level of expected return. This means, of course, that the general conclusions obtained from previous work with single-period utility of terminal wealth models regarding the portfolio decisions of risk-averse investors and the characteristics of general equilibrium remain valid when consumption and investment are jointly considered.

Since a Von Neuman–Morgenstern utility function is unique only up to a positive linear transformation, and since the return on the portfolio is $R_t = \Delta W_t / W_t = (W_{t+1}/W_t) - 1$, we can express the investor's consumption-portfolio problem as

$$\max_{x_i} E[U(C_t, R_t)] , \qquad (2.3)$$

and we assume U is monotone increasing and strictly concave in (C_t, R_t). (We state [2.3] in terms of R because it is more convenient and avoids problems with scale in making comparisons of portfolios later on.)

We have now set the foundation for consideration of the normative mean-variance portfolio models of Markowitz [40, 42] and Tobin [61, 62], which in turn provide much of the motivation for the Sharpe [52] and Lintner [37] models of general equilibrium conditions in the capital asset markets. As we shall see, these results provide the key to the solution of the portfolio evaluation problem.

Thus, let us turn to a brief review of the mean-variance portfolio models. Suffice it to say that all of these models are based on the existence of finite variances for the distributions of security returns. Empirical work by Mandelbrot [38], Fama [12], and Roll [46], however, indicates that the distributions of returns

[12] An asset is perfectly liquid if (*a*) at any particular time the buying and selling prices are identical and (*b*) any quantity can be bought or sold at this price. Thus, transactions costs are assumed to be equal to zero.

[13] In the empirical tests to come later, this may seem to be a significant restriction, for an investor in a high marginal income tax bracket will certainly not be indifferent to the form (capital gains or income dividends) in which he receives his returns. However, in practice this may not be as restrictive an assumption as we might believe. Horowitz [30], examining the properties of a model for ranking mutual funds, finds that the explicit allowance for differential tax rates on income and capital gains results in only minor effects on the relative rankings of ninety-eight funds. In choosing a portfolio for a particular investor, however, these tax considerations must be taken into account.

on common stocks and bonds seem to conform to the members of the Stable class of distributions for which the mean exists but the variance does not.[14] At this time we merely point out that Fama [13, 16] has demonstrated that with some modifications most of the results obtained for the special case of finite variances also extend to the more general case where the distributions on returns are allowed to be any symmetric member of the class of stable distributions with finite mean.

We continue the discussion in the mean-variance framework for the moment under the assumption that the probability distributions of all security returns have finite variances. The extension of the mean-variance results to a world characterized by distributions of returns with infinite variances will be considered in Section III-C below.

B. THE NORMATIVE THEORY OF PORTFOLIO ANALYSIS

The expected utility maxim and the diversification of investments.—Markowitz [40, 42] and Tobin [61, 62] have shown that diversification is the logical consequence for risk-averse investors whose objective function[15] can be written as

$$\max_{x_i} E[U(R_t)] . \qquad (2.4)$$

In particular, the utility maximizing portfolio for any investor will be a mean-variance efficient portfolio in the sense that it offers minimum variance for a given level of expected return and maximum expected return for a given level of variance if (1) the investor's utility function (2.4) meets the condition that $U' > 0$ and $U'' < 0$, and (2) the distributions of asset and portfolio returns are of the same form[16] and are completely described by two parameters (cf. Tobin [61, 62]). These conditions imply that all asset returns must be normally distributed for mean-variance efficiency to be meaningful.[17] In addition, Fama [16] has demonstrated that this theorem can be extended to the general class of symmetric Stable distributions with finite first moment (which, incidentally, seem to describe the empirical distributions of security returns quite well; cf. Fama [12], Mandelbrot [38], and Roll [46]). But we defer discussion of this point to Section III-C.

Figure 1 gives a geometric presentation of the Markowitz mean-variance model. Letting $\sigma(R)$ be the standard deviation of future return, the shaded area in Figure 1 represents all possible combinations of risk and return available from investments in risk-bearing securities. The portfolios lying on the boundary *ABCD* represent the set of mean standard deviation (or mean variance) efficient portfolios, since they all repre-

[14] It may be noted here that the Gaussian or normal distribution is the special case of this class of distributions with characteristic exponent $\alpha = 2$.

[15] By Fama's results [15, 16] we have seen that the conclusions drawn from an investigation of the implications of (2.4) also hold for the solution to (2.1). Thus, for simplicity, from this point on we shall ignore the consumption decision, C_t, and couch our discussion in terms of the single-period utility of return function given by (2.4).

[16] This qualification is extremely important and often overlooked. Samuelson [48] presents a simple example of a two-parameter distribution for which the analysis fails for precisely this reason. It is also interesting to note that the Stable class of distributions (cf. Feller [23, chap. xvii]) are the only distributions which are stable under addition. That is, Stable distributions are the only distributions for which weighted sums of random variables (i.e., a portfolio) will have the same form as the underlying random variables. But this means the only distribution for which the mean-variance version of the Tobin theorem holds is the normal (cf. Fama [16]).

[17] Contrary to generally accepted opinion, the assumption of quadratic utility functions is not sufficient to guarantee that the utility maximizing portfolio will be mean-variance efficient (cf. Borch [5, p. 20–21]). Thus, the conditions given in the text are the only conditions which will justify the mean-variance framework.

sent possible investments yielding maximum expected returns for given risk and minimum risk for given expected returns.

As Tobin [61] has shown, the normality of security returns and the existence of risk aversion on the part of the investor are sufficient to yield a family of positively sloping convex indifference curves (represented by I_1, I_2, I_3) in the mean standard deviation plane of Figure 1. The shaded area in Figure 1 represents the opportunity set available to the investor in the absence of a riskless asset, and the boundary of this set *ABCD* represents the set of efficient portfolios in the Markowitz sense. An investor limited only to investments in *risky assets* who has the particular indifference map shown in Figure 1 will maximize his expected future utility with an investment in portfolio B, yielding $E(R_B)$ and $\sigma(R_B)$ with utility I_1.

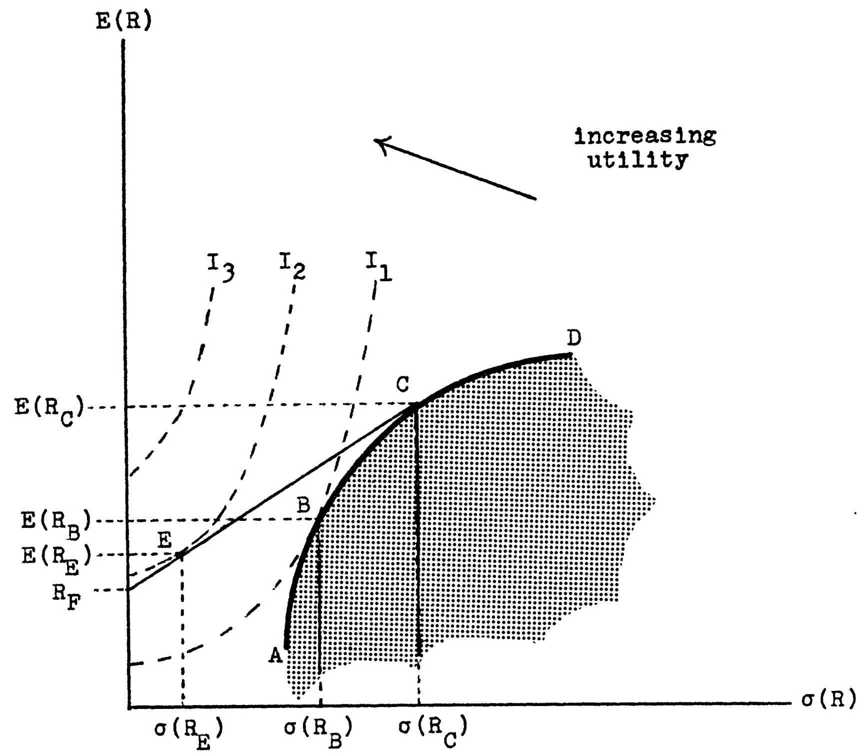

FIG. 1.—The maximization of investor utility given the existence of a risk-free asset

Implications of the existence of a riskless asset.—Portfolio B portrayed in Figure 1 represents an optimal solution to the portfolio problem only in the case where investment is restricted to risky assets. Let us now assume the existence of a risk-free asset F, yielding a certain future return R_F as drawn in Figure 1.[18]

[18] Such an instrument might be cash (yielding no positive monetary return), an insured savings account, or a non-coupon-bearing government bond having a maturity date coincident to the investor's horizon date. In the latter case, of course, the investor can be assured of realizing the yield to maturity with certainty if he holds the bond to maturity. Since we have assumed the investor will not change his portfolio in the interim period, any intermediate fluctuations in price do not present him with risk. We are ignoring the problems associated with changes in the general price level and shall continue do so in the remainder of the paper.

An investor faced with the possibility of an investment in such a risk-free asset, as well as in a risky asset, can construct a portfolio of the two assets which will allow him to reach any combination of risk and return lying along a straight line connecting the two assets in the mean standard deviation plane (cf. Tobin [61]). Clearly, all portfolios lying below point C along $ABCD$ in Figure 1 are inefficient, since any point on the line $R_F C$ given by

$$E(R) = R_F + \frac{E(R_C) - R_F}{\sigma(R_C)} \cdot \sigma(R) \qquad (2.5)$$

$$\sigma(R) < \sigma(R_C)$$

represents a feasible solution. Thus, the investor may distribute his funds between portfolio C and security F such that his combined portfolio, call it E, yields him $E(R_E)$, $\sigma(R_E)$, and maximum utility of $I_2 > I_1$. In addition, if the investor can borrow as well as lend at the riskless rate R_F, the set of feasible portfolios represented by the line $R_F C$ and equation (2.5) extends beyond point C.

C. A THEORY OF CAPITAL ASSET PRICES

Sharpe [52], Lintner [36, 37], and Mossin [43] starting with the normative models of Markowitz and Tobin have developed similar theories of capital market equilibrium under conditions of risk. The following assumptions underlie all three models: (1) all investors have identical horizon periods; (2) all investors may borrow as well as lend funds at the riskless rate of interest; and (3) investors have homogeneous expectations regarding expected future return and standard deviation of return on all assets and all covariances of returns among all assets. Sharpe observed that investors would attempt to purchase *only* those assets in portfolio C and the riskless security F of Figure 1. Thus, we have a situation in which the market for capital assets would be out of equilibrium unless C is the "market portfolio," that is, a portfolio which contains every asset exactly in proportion to that asset's fraction of the total value of all assets. Conceptually, if the market were out of equilibrium the prices of assets in C would be instantaneously bid up and the prices of assets not in C would fall until such time as all assets were held.

In equilibrium, all investors who select ex ante efficient portfolios will have mean standard deviation combinations which lie along the line $R_F Q$ in Figure 2, their individual location determined by their degree of risk aversion. Sharpe [52] has asserted that in equilibrium the efficient set may be tangent to $R_F Q$ at multiple points as in Figure 2. However, whether or not this ever occurs,[19] the market portfolio M must always be one of the tangency points (cf. Fama [14] and Fama and Miller [20]).

Most important, however, is the result that in equilibrium the expected return on any *efficient portfolio* ϵ will be linearly related to the expected return on the market portfolio in the following manner:[20]

$$E[R_\epsilon | E(R_M), \sigma(R_\epsilon)] = R_F + \frac{E(R_M) - R_F}{\sigma(R_M)} \sigma(R_\epsilon) \,. \qquad (2.6)$$

The concept of systematic risk.—In addition, Sharpe, Lintner, and Mossin

[19] While it is possible for multiple tangency points to exist, it is highly improbable that this would ever occur. The existence of multiple tangency points would require that the returns on one or more individual securities were perfectly correlated with those of the market portfolio M.

[20] For reasons which will become clear below, we choose to write equations like (2.6) (and [2.7] below) recognizing only two of the conditioning variables explicitly on the left-hand side. These variables become crucial to distinctions we wish to maintain below.

have shown that if the capital market is in equilibrium the expected return on any *individual security* (or portfolio) will be a linear function of the covariance of its returns with that of the market portfolio.[21] The function is:

$$E[R_j | E(R_M), \text{cov}(R_j, R_M)] = R_F + \frac{E(R_M) - R_F}{\sigma^2(R_M)} \text{cov}(R_j, R_M) . \quad (2.7)$$

diversified portfolios and can assume that the capital markets are in equilibrium, (2.7) implies that the relevant measure of the riskiness of any security (or portfolio) is the quantity $\text{cov}(R_j, R_M)$, and the market price per unit of risk is $[E(R_M) - R_F]/\sigma^2(R_M)$.

We shall see in Section III that this result, (2.7), will become the foundation of the portfolio evaluation model discussed there. Thus, a detailed discussion of the implications of (2.7) is given in Section III.

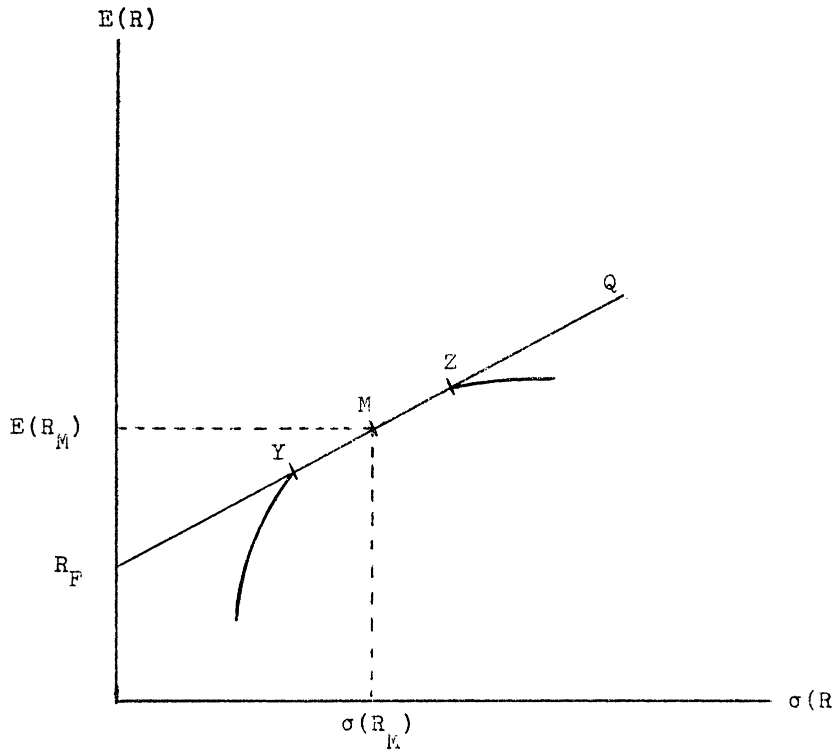

FIG. 2.—Possible configuration of the "efficient set," given equilibrium in the capital markets

It is important to note that (2.6) holds *only* for efficient portfolios and (2.7) holds for any individual security or any portfolio regardless of whether it is efficient. Thus, as long as we are concerned with risk in the context of efficiently

III. THE SINGLE-PERIOD HOMOGENEOUS HORIZON MODEL

The reader will recall that one of the assumptions made in the derivation of the asset pricing model was that all investors are one-period expected utility

[21] We note here that Jack Treynor also had independently arrived at these results at about the same time as Sharpe and Lintner. Unfortunately, his excellent work [64] remains unpublished.

maximizers having a common horizon date. The assumption of identical investor decision horizons is admittedly unrealistic, but for the moment we proceed with the development of the model within this context. It will be shown in Section IV that the asset pricing model and the portfolio evaluation model based on it can be extended to a world in which investors have horizon periods of differing lengths and trading of assets is allowed to take place continuously through time.

A. A STANDARD OF COMPARISON

A major problem encountered in developing a portfolio evaluation model is the establishment of a norm or standard for use as a bench mark. The discussion of Section II points to a natural standard—the performance of the market portfolio, M. As long as the market is in equilibrium we know that ex ante this portfolio must be a member of the efficient set. Ex post, of course, this portfolio will not dominate all others, since in a stochastic model such as this, realized returns will seldom be equal to expectations.

The market portfolio also offers another interpretation as a standard of comparison, since it represents the results which could have been realized (ignoring transaction costs) by one particular naïve investment strategy, that is, purchasing each security in the market in proportion to its share of the total value of all securities.

Thus, the concept of the market portfolio provides a natural point of comparison. However, as mentioned earlier, we cannot compare returns on portfolios with differing degrees of risk to the same standard; but this problem may be resolved by reference to the asset pricing model discussed in Section II. Recall that the Sharpe-Lintner asset pricing model indicates that the expected return on any asset (or portfolio of assets) is given by (2.7)

$$E\left[R_j | E(R_M), \frac{\text{cov}(R_j,R_M)}{\sigma^2(R_M)}\right] = R_F + [E(R_M) - R_F]\frac{\text{cov}(R_j,R_M)}{\sigma^2(R_M)} . \tag{2.7}$$

Let us define

$$\beta_{1j} = \frac{\text{cov}(R_j,R_M)}{\sigma^2(R_M)} \tag{3.1}$$

so that we now measure the risk of any security[22] j relative to the risk of the market portfolio. (The term β_{1j} will henceforth be referred to as the "systematic" risk of the jth asset or portfolio, and the first subscript, here 1, will be used to distinguish between three alternative interpretations of the coefficients.) Thus, if the asset pricing model is valid and the capital market is in equilibrium, the expected one-period return on any asset (or portfolio of assets) will be a linear function of the quantity β_1 as portrayed in Figure 3. The point M represents the expected return and systematic risk of the market portfolio, and the point R_F represents the return on the risk-free asset. Since we are measuring risk relative to the risk of the market portfolio, it is obvious that the risk of the market portfolio is unity, since

$$\beta_{1M} = \text{cov}(R_M,R_M)/\sigma^2(R_M) = \sigma^2(R_M)/\sigma^2(R_M) = 1 .$$

Thus, conditional on the expected returns on the market portfolio and the risk-free rate, (2.7) gives us the relationship between the expected returns on any asset (or collection of assets) and its level of systematic risk β_{1j}. However,

[22] Henceforth, we shall use the terms "asset" and "security" interchangeably.

since expectations can be observed only with error, these results will be much more useful if they can be translated into a relationship between ex post realizations.[23] We now show how this may be accomplished by utilizing the additional structure imposed on the asset pricing model by the assumptions of what Blume [4] and Fama [14, 16] have called the "market model." In Part B we consider the model under the assumption that the distributions of returns are normal and in Part C under the assumption that the distributions of returns conform to the infinite variance (but finite mean) members of the symmetric Stable family of distributions.

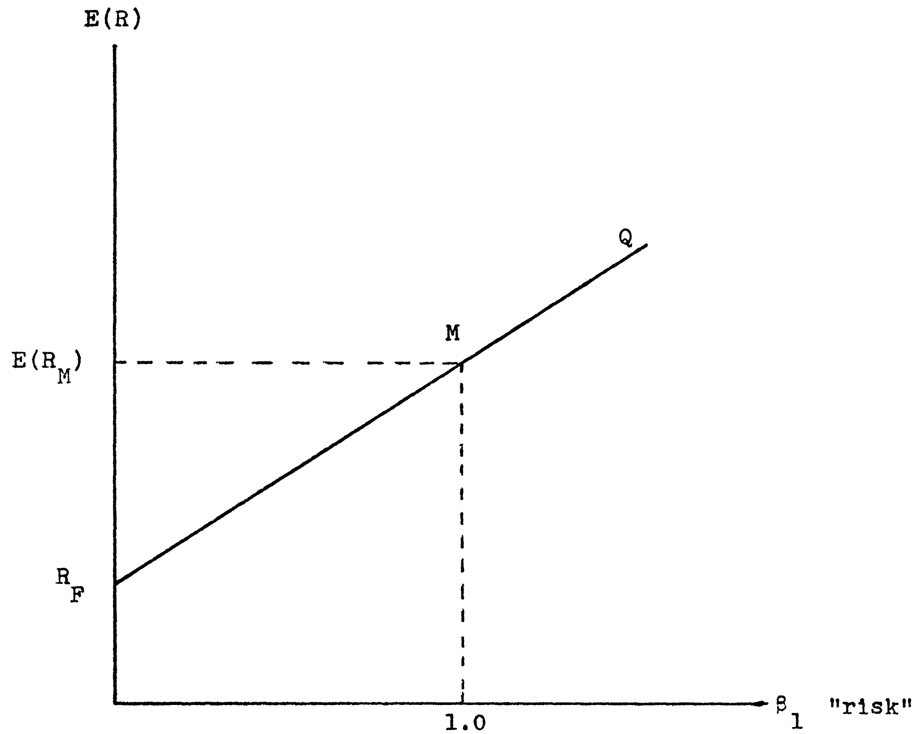

FIG. 3.—The relationship between the expected return on any asset (or collection of assets) and systematic risk (β_1) as implied by the capital asset pricing model.

[23] For a discussion of the problems and issues which can arise around just this question regarding ex ante relationships, see West [66] and Sharpe [58]. While the criticisms raised by West are legitimate, we shall see below that the problems can be completely surmounted in that we can derive explicit relationships between ex post variables which still yield testable results. This same issue also arises in the debate contained in [8] and [28].

B. SYSTEMATIC RISK IN THE CONTEXT OF THE GAUSSIAN MARKET MODEL

The model.—The market model was originally suggested by Markowitz [42, p. 100] and analyzed in considerable detail by Sharpe [51, 52, 57], who referred to it as the "diagonal model." Simply stated, the model postulates a linear relationship between the returns on any security and a general "market factor."[24]

[24] The model described by equations (3.2) and (3.3) is slightly different from the diagonal model originally proposed by Markowitz, analyzed by

That is, we express the returns on the jth security as

$$R_j = E(R_j) + b_j\pi + e_j \qquad j = 1, 2 \ldots, N, \tag{3.2}$$

where the "market factor" π is defined such that $E(\pi) = 0$, b_j is a constant, π and e_j are all normally distributed random variables, and N is the total number of securities in the market. The following assumptions are made regarding the disturbance terms e_j:

$$E(e_j) = 0 \qquad j = 1, 2, \ldots, N \tag{3.3a}$$

$$E(e_j\pi) = 0 \qquad j = 1, 2, \ldots, N \tag{3.3b}$$

$$E(e_je_i) = \begin{cases} 0 & i \neq j \\ \sigma^2(e_j) & i = j. \end{cases} \tag{3.3c}$$

Now let V_j be the total value of all units of the jth security outstanding. Then

$$x_j = V_j \Big/ \sum_{i=1}^{N} V_i$$

is the fraction of the jth security in the market portfolio defined earlier. The returns on the market portfolio R_M are given by

$$R_M = \sum_j x_jR_j = \sum_j x_jE(R_j) + \sum_j x_jb_j\pi + \sum_j x_je_j. \tag{3.4}$$

As Blume [4] and Fama [16] have pointed out, the market factor π is unique up to a linear transformation, and thus we can always change the scale of π such that

$$\sum_j x_jb_j = 1.$$ [25]

Hence, with no loss of generality, we assume this transformation and reduce (3.4) to

$$R_M = E(R_M) + \pi + \sum_j x_je_j. \tag{3.5}$$

Now we saw earlier that the measure of systematic risk is $\text{cov}(R_j, R_M)$. By direct substitution from (3.2) and (3.5) into the definition of the covariance,

$$\begin{aligned} \text{cov}\,(R_j,R_M) &= \text{cov}\left\{\left[E(R_j) + b_j\pi + e_j,\right]\left[E(R_M) + \pi + \sum_i x_ie_i\right]\right\} \\ &= b_j\sigma^2(\pi) + x_j\sigma^2(e_j) \qquad j = 1, 2, \ldots, N \end{aligned} \tag{3.6}$$

and

$$\sigma^2(R_M) = \sigma^2(\pi) + \sum_j x_j^2\sigma^2(e_j). \tag{3.7}$$

Sharpe, and empirically tested by Blume [4]. The model is

$$R_j = A_j + B_j I + u_j, \tag{3.2a}$$

where I is some index of market returns, u_j is a random variable uncorrelated with I, and A_j and B_j are constants. The differences in specification in (3.2) are necessary in order to avoid the overspecification pointed out by Fama [14] which arises if one chooses to interpret the market index I as an average of security returns or as the returns on the market portfolio M (cf. Lintner [37] and Sharpe [52, 56]). That is, if I is some average of security returns, then the assumption that u_j is uncorrelated with I cannot hold, since I contains u_j.

[25] Reproducing Fama's argument directly, if we have the untransformed market factor π^* and

$$\sum_j x_jb_j^* \neq 1,$$

where the b_j^* are defined by $R_j = E(R_j) + b_j^*\pi^* + e_j$, we can create

$$\pi = \pi^* \sum_j x_jb_j^*.$$

Now $R_j = E(R_j) + b_j\pi + e_j$, where

$$b_j = b_j^* \Big/ \sum_i x_ib_i^*$$

and

$$\sum_j x_jb_j = 1.$$

Hence, restating the results of the capital asset pricing model given in (2.7) in terms of the parameters of the market model, we have[26]

$$E[R_j|E(R_M),(\cdot)] = R_F + [E(R_M) - R_F] \times \left[\frac{b_j\sigma^2(\pi) + x_j\sigma^2(e_j)}{\sigma^2(R_M)}\right], \quad (3.8)$$

where $(\cdot)$ refers to the arguments in brackets on the RHS of (3.8). Now define

$$\beta_{2j} = \frac{b_j\sigma^2(\pi) + x_j\sigma^2(e_j)}{\sigma^2(R_M)}, \quad (3.9)$$

which is the measure of systematic risk in the context of the Gaussian market model. All previous discussion regarding the interpretation of β_{1j} also applies to β_{2j}.

However, (3.9) can be considerably simplified by noting that we can invoke several approximations and thereby eliminate the strictly unobservable market factor π from the expression. The results of King[27] [34] and Blume[28] [4] imply that the market factor π accounts for approximately 50 per cent of the variability of individual security returns.[29] Since $\sigma^2(R_j) = b_j^2\sigma^2(\pi) + \sigma^2(e_j)$, and since the average b_j is equal to unity, the results of King and Blume imply that $\sigma^2(e_j)$ is roughly the same order of magnitude as $\sigma^2(\pi)$.

Let us examine the expression for $\sigma^2(R_M)$ in light of these facts. The last term on the RHS of (3.7) can be approximately expressed as

$$\sum_j x_j^2\sigma^2(e_j) \cong \frac{N}{N^2}\overline{\sigma^2(e)} = \frac{1}{N}\overline{\sigma^2(e)} \quad (3.10)$$

where $\overline{\sigma^2(e)}$ is the average variance of the disturbance terms. Recall that since x_j is the ratio of the value of the jth security to the total value of all securities, it must on the average be on the order of $1/N$, where N is the total number of distinct securities in the market. Since there are more than 1,000 securities on the New York Stock Exchange alone, x_j will be much smaller than 1/1000 on the average,[30] and thus (3.10) will be minute relative to $\sigma^2(\pi)$. Hence,

$$\sigma^2(R_M) \cong \sigma^2(\pi) . \quad (3.11)$$

Substituting for $\sigma^2(\pi)$ in (3.9), we have

$$\beta_{2j} \cong \frac{b_j\sigma^2(R_M) + x_j\sigma^2(e_j)}{\sigma^2(R_M)} = b_j + \frac{x_j\sigma^2(e_j)}{\sigma^2(R_M)} . \quad (3.12)$$

[26] This is essentially the same expression as Lintner [37] arrived at, but as we have seen, and as Fama [14] has already shown, the results of Sharpe originally stated in (2.7) are in no way inconsistent with (3.8).

[27] King examined sixty-three securities in the period June, 1927, to December, 1960, by methods of factor analysis. He found that the market factor on the average accounts for approximately 50 per cent of the variability of the monthly returns on the individual securities, and various industry factors account for another 10 per cent. We have ignored these industry factors in constructing the model, since they are relatively unimportant and their inclusion would introduce a great deal of additional complexity.

[28] Blume, using regression analysis, also finds that a market index accounts for an average of 50 per cent of the variability of the monthly returns on 251 securities in the period January, 1927, to December, 1960.

[29] There is some indication in Blume's results, however, that this proportion may be declining in recent times.

[30] There are some firms, of course, for which x_j is much larger than 1/1000. Data obtained from Standard & Poor's indicates that as of December 31, 1964, the four largest firms on the New York Stock Exchange and their percentages of the total values of the Standard & Poor Composite 500 Index were: A.T. & T., 9.1 per cent; General Motors, 7.3 per cent; I.B.M., 3.7 per cent; and DuPont, 2.9 per cent. Thus, the largest value that the fraction x_j could take in 1964 was .091, and even this is an overstatement, since the 500 securities were obviously not the total universe—which, of course, includes all other exchanges, unlisted securities, and debt instruments as well.

For simplicity, let us define

$$z_j = \frac{x_j \sigma^2(e_j)}{\sigma^2(R_M)} \,. \qquad (3.13)$$

Substituting for $E(R_j)$ from (3.9) and (3.8) into (3.2), we have

$$\begin{aligned} R_j &= E(R_j) + b_j \pi + e_j \\ &= R_F(1 - \beta_{2j}) + \beta_{2j} E(R_M) \qquad (3.14) \\ &\qquad + b_j \pi + e_j \,. \end{aligned}$$

Adding and subtracting $z_j \pi$ and

$$\beta_{2j} \sum_i x_i e_i$$

on the RHS of (3.14) gives

$$\begin{aligned} R_j = R_F(1 - \beta_{2j}) &+ \beta_{2j} E(R_M) \\ &+ b_j \pi + z_j \pi + \beta_{2j} \sum_i x_i e_i \qquad (3.15) \\ &- z_j \pi - \beta_{2j} \sum_i x_i e_i + e_j \,. \end{aligned}$$

Noting that $\beta_{2j} \cong b_j + z_j$, using the definition of R_M from (3.5) and simplifying, we get

$$\begin{aligned} R_j = R_F(1 - \beta_{2j}) &+ R_M \beta_{2j} \\ &- z_j \pi - \beta_{2j} \sum_i x_i e_i + e_j \,. \qquad (3.16) \end{aligned}$$

Now we have an ex post relationship in which all the important variables are measurable.[31] By assumption (3.3*a*), $E(e_j) = 0$. Thus, eliminating $z_j \pi$ and

$$\beta_{2j} \sum_j x_i e_i$$

from (3.16) by the arguments of note 31 above, we see that to a very close approximation the *conditional* expected return on the *j*th security is given by

$$\begin{aligned} E(R_j | R_M, \beta_{2j}) \cong R_F(1 - \beta_{2j}) & \\ + R_M \beta_{2j} \,. & \qquad (3.17) \end{aligned}$$

Equation (3.17) is an important result. It gives us an expression for the expected return on security j conditional on the ex post realization of the return on the market portfolio.[32] Recall that equation (2.7), the result of the capital asset pricing model, provides only an expression for the expected return on the *j*th security conditional on the ex ante expectation of the return on the market portfolio. This result (eq. [3.17]) becomes extremely important in considering the empirical application of the model.[33] We now have shown that we can explicitly use the observed *realization* of the return on the market portfolio without worrying about using it as a proxy

[31] Note that $z_j \pi$ will be trivially small, since by our previous arguments $\sigma^2(e_j) \cong \sigma^2(\pi) \cong \sigma^2(R_M)$ and x_j is on the average less than 1/1000. Thus,

$$z_j \pi = \frac{x_j \sigma^2(e_j)}{\sigma^2(R_M)} \pi \cong x_j \pi \cong \frac{1}{1000} \pi$$

and is unimportant.

Note that

$$\beta_{2j} \sum_i x_i e_i$$

will be unimportant also, since by assumption the e_j are independently distributed random variables with $E(e_j) = 0$. We have already seen that the variance of this weighted average (given by [3.10]) will be minute. But since

$$E\left(\sum_i x_i e_i \right) = \sum_i x_i E(e_i) = 0 \,,$$

and its variance is extremely small, it is unlikely that it will be very different from zero at any given time.

[32] Of course, as far as the algebraic manipulations are concerned, we do not need the market model to get this result. However, the implications of the results derived in the absence of the market model are not consistent with the observed behavior of the world. That is, consider the formulation in which π always equals zero. The ex post returns on the market portfolio would be given by

$$R_M = E(R_M) + \sum_i x_i e_i \,.$$

But in the discussion above, we saw that the last term has expectation equal to zero and an infinitesimal variance. Thus, this formulation implies that the realized returns on the market portfolio would never differ from the expected returns by any amount of consequence—a result clearly contradicted by the behavior of real world prices.

[33] See, for example, the discussions in references 8, 28, 58, and 66 regarding the problems associated with testing models stated in terms of ex ante relationships on ex post empirical data.

for the expected return and without worrying about devising an ad hoc expectations-generating scheme.

The measure of portfolio performance in the context of Gaussian distributions.—Using equations (3.16) and (3.17), we can now define an ex post measure of portfolio performance as

$$\begin{aligned}\delta_{2j} &= R_j - E(R_j | R_M, \beta_{2j}) \\ &= R_j - [R_F(1 - \beta_{2j}) + R_M\beta_{2j}] \quad (3.18) \\ &= z_j\pi - \beta_{2j}\sum_i x_i e_i + e_j \, .\end{aligned}$$

But by our previous arguments, the quantity $z_j\pi$ will be minute. In addition, the likelihood of $\beta_{2j}\Sigma x_i e_i$ ever being much different from zero is extremely small, since its expected value is equal to zero and its variance is close to zero (cf. n.31 above). By these arguments, we may ignore these terms in (3.18), and we have to a close approximation

$$\delta_{2j} \cong e_j \, . \quad (3.19)$$

Figure 4 gives a geometric presentation of these concepts. The point M represents the *realized* returns on the market portfolio, and of course its systematic risk (plotted on the abscissa) is unity.[34] The point R_F represents the returns on the risk-free asset, and the equation of the line R_FMQ is

$$E(R | R_M, \beta_2) = R_F + (R_M - R_F)\beta_2 \, . \quad (3.20)$$

Let the point i represent the ex post returns R_i on any portfolio i and let β_{2i} be its level of systematic risk. Then the vertical distance between the risk-return combination of any portfolio i and the line R_FMQ in Figure 4 is our measure of the performance of portfolio i.

The measure δ_2 may also be interpreted in the following manner: Let FM_i be a portfolio consisting of a combined investment in the risk-free asset F and the market portfolio M offering the same degree of risk β_{2i} as the portfolio i. Now δ_{2i} may be interpreted as the difference in return realized on the ith portfolio and the return R_{FM_i} which could have been earned on the equivalent risk market portfolio FM_i. If $\delta_{2i} \geq 0$, the portfolio i has yielded the investor a return greater than or equal to the return on a combined investment in M and F with an identical level of systematic risk. It should be noted that since (3.18) is stated in terms of the *observed* return on the market portfolio, the performance measure δ_{2i} allows for the actual relationship between risk and return which existed during the particular holding period examined.

A discussion of the criteria to be used in judging a portfolio's performance will be postponed until Section V, at which time the entire model will have been developed. Meanwhile, in the next section we shall consider the extension of the model to a world in which the distributions of security returns are non-Gaussian members of the Stable class.

C. SYSTEMATIC RISK AND THE STABLE MARKET MODEL

The model.—As mentioned earlier, there is considerable empirical evidence (Fama [12], Mandelbrot [38], Roll [46]) indicating that distributions of security returns conform to the members of the Stable class of distributions which have finite means but infinite variances. However, Fama [13] has shown that the market model can be used to develop a portfolio model analogous to the mean-variance models of Markowitz, Tobin, and Sharpe in the context of a market in which returns are generated by non-

[34] Note that β_2 merely represents the specific expression for risk in the context of the infinite variance market model and will be defined below.

Gaussian finite mean members of the Stable family of distributions. Moreover, Fama [16] has also demonstrated that the Sharpe-Lintner capital asset pricing models can be generalized to a market characterized by returns with infinite variance distributions. The following discussion draws heavily on his extension of the asset pricing model. The reader is referred to Fama [16] for proofs. We begin with a few brief comments on the parameters of Stable distributions.[35]

Stable distributions have four parameters, α, β, δ, and γ. The parameter α is called the characteristic exponent and has range $0 < \alpha \leq 2$. The special case of the Stable distribution with $\alpha = 2$ is the Gaussian or normal distribution and is the only distribution with finite second- and higher-order moments.

The parameter β with range $-1 \leq$

[35] For a much more complete description of properties of Stable distributions, see the Appendix in Fama [12] and the references therein.

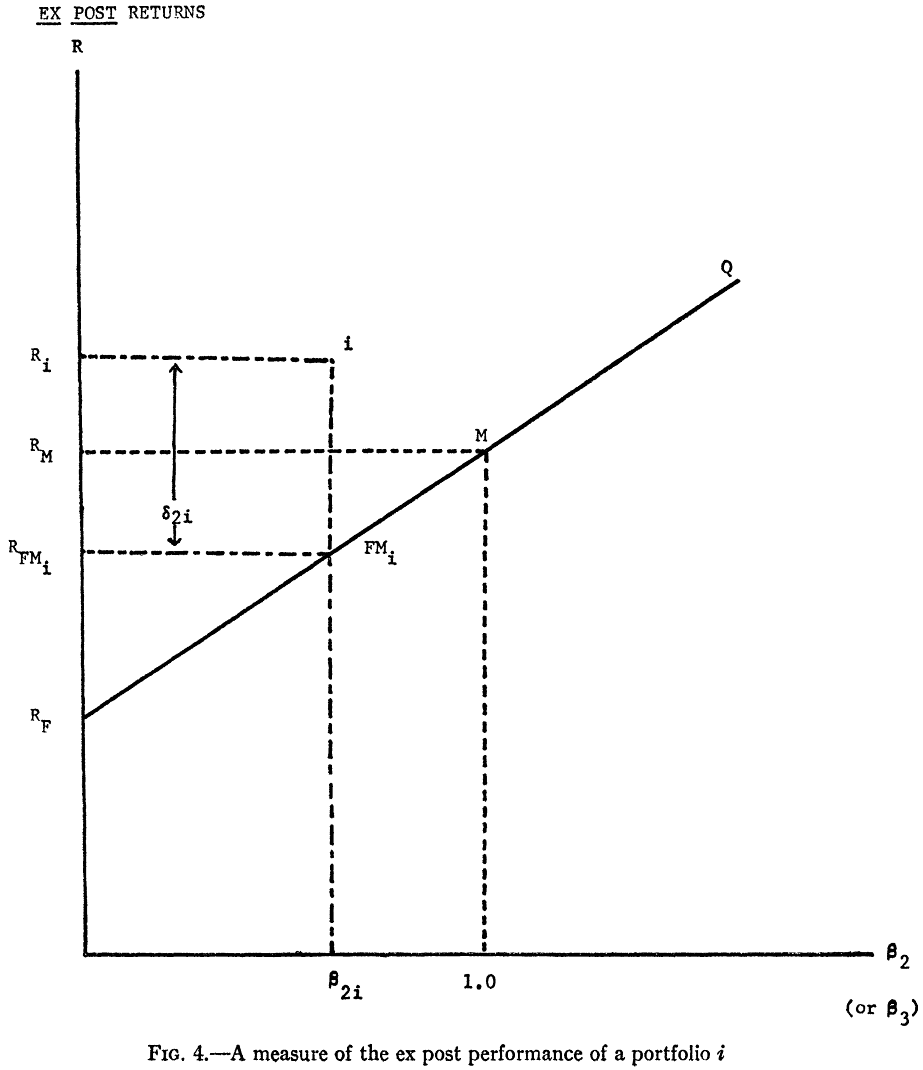

FIG. 4.—A measure of the ex post performance of a portfolio *i*

$\beta \leq 1$ determines the skewness of the distribution. When $\beta = 0$ the distribution is symmetric, when $\beta > 0$ the distribution is skewed left, and when $\beta < 0$ the distribution is skewed right. We assume in the following discussion (as does Fama) that we are dealing with symmetric distributions,[36] and therefore $\beta = 0$.

The parameter δ is the location parameter of the distribution, and for distributions with $\beta = 0$ and $1 < \alpha \leq 2$, δ is the expected value or mean. For distributions with $0 < \alpha \leq 1$, the mean does not exist, but for distributions with $\beta = 0$, δ is the median. As Fama [13] has shown, diversification is meaningless in a market characterized by distributions with $\alpha \leq 1$. In addition, Fama [12] and Roll [46] find that estimates of the characteristic exponent α for common stocks and U.S. Treasury bills indicate $\alpha \geq 1$. Thus, we also assume in the following that $1 < \alpha \leq 2$ and therefore $\delta(R) = E(R)$.

The final parameter $\gamma(\gamma > 0)$ defines the scale or dispersion of the Stable distribution. For the Gaussian distribution with $\alpha = 2$, $\gamma = \frac{1}{2}\sigma^2$ where σ^2 is the variance. Unfortunately, as mentioned earlier, when $\alpha < 2$ the variance does not exist and analytical solutions for the exact definition of γ are known only for several special cases; for example, for the Cauchy case ($\alpha = 1$), γ is exactly equal to the semi-interquartile range.[37] Fama and Roll [21] have demonstrated that for α in the range $1 < \alpha \leq 2$, γ corresponds approximately to the semi-interquartile range raised to the α power.

The Stable market model again consists of equation (3.2):

$$R_j = E(R_j) + b_j\pi + e_j \qquad j = 1, 2, \ldots, N \tag{3.2}$$

with all the variables defined as before. However, in place of (3.3) it is now assumed that π and e_j ($j = 1, 2, \ldots, N$) are independently distributed symmetric Stable variables all having the same characteristic exponent α, ($1 < \alpha \leq 2$). The location parameters of π and e_j are, respectively, $\delta(\pi) = E(\pi) = 0$, $\delta(e_j) = E(e_j) = 0$ ($j = 1, 2, \ldots, N$), and their dispersion parameters are $\gamma(\pi)$ and $\gamma(e_j)$ ($j = 1, 2, \ldots, N$). Under these conditions, the location parameter of R_j is $E(R_j)$ and the scale parameter of R_j is given by

$$\gamma(R_j) = \gamma(\pi)|b_j|^\alpha + \gamma(e_j) . \tag{3.21}$$

By the same arguments as in the finite variance case, the return on the market portfolio is given by

$$R_M = E(R_M) + \pi + \sum_j x_j e_j \tag{3.5}$$

and the scale parameter of the distribution of the returns on the market portfolio is

$$\gamma(R_M) = \gamma(\pi) + \sum_j \gamma(e_j)|x_j|^\alpha . \tag{3.22}$$

More significantly, Fama has demonstrated (by arguments directly analogous to those of Sharpe and Lintner presented earlier) that, given the assumptions of the Stable market model (and the previously stated assumptions necessary to the Sharpe-Lintner model), the expected return on any security j will be given by

$$E(R_j) = R_F + [E(R_M) - R_F] \times \left[\frac{\gamma(\pi)b_j + \gamma(e_j)|x_j|^{\alpha-1}}{\gamma(R_M)}\right] . \tag{3.23}$$

[36] The assumption of symmetry seems to be satisfied quite well by the empirical distributions of security returns stated in terms of continuously compounded rates. Furthermore, we shall see in Section IV that the solution to the "horizon" problem implies that all returns must be measured as continuously compounded rates in order for the model to hold. Thus, the assumption of symmetry seems quite appealing.

[37] Defined as half the difference between the .75 and .25 fractiles.

Equation (3.23) is directly comparable to the results given in (3.8), which were arrived at under the assumption of finite variances. (Note that in the case $\alpha = 2$, [3.23] reduces directly to [3.8].)

Now define

$$\beta_{3j} = \frac{\gamma(\pi)b_j + \gamma(e_j)|x_j|^{\alpha-1}}{\gamma(R_M)}, \quad (3.24)$$

which is the measure of systematic risk in the context of the Stable market model. As before, all previous discussion regarding β_{1j} and β_{2j} also applies to β_{3j}.

Hence, we see that by making use of the characteristics of the market model, the capital asset pricing model can be extended to the case of infinite variance distributions where the concept of a covariance is undefined.

However, as in the finite variance case (and at the expense again of some degree of approximation), the expression for systematic risk (3.24) can be considerably simplified. As before, the results of King [34] and Blume [4] indicate that on the average the terms $\gamma(\pi)$ and $\gamma(e_j)$ are of about equal size. Likewise as before, the average x_j is on the order of $1/N$, N being very large. Hence, the last term on the RHS of (3.22) is approximately equal to

$$\sum_j \gamma(e_j)|x_j|^\alpha \cong \frac{N}{N^\alpha}\overline{\gamma(e)} = \frac{1}{N^{\alpha-1}}\overline{\gamma(e)} \quad (3.25)$$

where $\overline{\gamma(e)}$ is the average scale parameter of the disturbance terms. But since we have assumed $\alpha > 1$ and since empirical evidence (cf. Fama [12]) indicates $1.6 \leq \alpha \leq 1.9$, this term will be small relative to $\gamma(\pi)$. Thus,

$$\gamma(R_M) \cong \gamma(\pi), \quad (3.26)$$

and substituting for $\gamma(\pi)$ in (3.24), we have

$$\beta_{3j} \cong b_j + \frac{\gamma(e_j)|x_j|^{\alpha-1}}{\gamma(R_M)}. \quad (3.27)$$

Letting

$$v_j = \frac{\gamma(e_j)|x_j|^{\alpha-1}}{\gamma(R_M)},$$

we may transform (3.23) from an ex ante relationship into an ex post relationship by arguments identical to those for the Gaussian case examined earlier. The result is

$$R_j = R_F(1 - \beta_{3j}) + R_M\beta_{3j} - v_j\pi - \beta_{3j}\sum_i x_i e_i + e_j. \quad (3.28)$$

Similarly, we also define the analogous conditional expected return on the jth security (or portfolio) as

$$E(R_j|R_M,\beta_{3j}) = R_F(1 - \beta_{3j}) + R_M\beta_{3j}. \quad (3.29)$$

The measure of portfolio performance in the context of non-Gaussian Stable distributions.—The measure of portfolio performance in the context of infinite variance Stable distributions is directly analogous to the finite variance situation and is given by

$$\begin{aligned}\delta_{3j} &= R_j - E(R_j|R_M,\beta_{3j}) \\ &= R_j - [R_F(1 - \beta_{3j}) + R_M\beta_{3j}] \quad (3.30) \\ &= -v\pi - \beta_{3j}\sum_i x_i e_i + e_j.\end{aligned}$$

Again, our previous arguments indicate that $v_j\pi$ and

$$\beta_{3j}\sum_i x_i e_i$$

will be extremely small and hence can be ignored, leaving the result

$$\delta_{3j} \cong e_j. \quad (3.31)$$

Since the purpose of all the above has been to arrive at a measure of performance, we shall consider the quantities δ_{2j} and δ_{3j} very closely in determining criteria for judging the performance of a portfolio. Our goal is to arrive at criteria for judging a portfolio's performance to be *superior*, *neutral*, or *inferior*.[38] Given the stochastic nature of the model, it is not surprising that this becomes a probabilistic problem. However, in view of the fact that we are still working within the context of a single-period model, consideration of these questions will be postponed until we have considered the multiperiod model in Section IV.

IV. THE MULTIPERIOD HETEROGENEOUS HORIZON MODEL

A. THE HORIZON PROBLEM

The reader will recall that, in deriving the results of the capital asset pricing model given in (2.7), (3.8), or (3.23), it was assumed that all investors had horizon periods of identical length. This implies, of course, that all trading in the market takes place only at the beginning and end of this horizon period. The question we now face is whether this theory will also apply to a market in which trading takes place almost continuously and in which investors most certainly have different (and overlapping) horizon periods.

The difficulty caused by unequal horizon intervals among investors is crucial to the portfolio evaluation problem, since we want to be able to evaluate a portfolio's performance over any horizon interval. But if equations (2.7), (3.8), and (3.23) hold only for a particular discrete horizon interval, then equations (3.18) and (3.30) defining a measure of portfolio performance also hold only for that horizon interval. Furthermore, consideration of this horizon problem[39] leads to other perplexing questions. For instance, is the existence of a discrete horizon interval consistent with a world in which trading takes place almost continuously? If so, may we arbitrarily choose the beginning of the portfolio evaluation period (i.e., the beginning of the horizon interval) to be any point in calendar time? How do we go about estimating the length of the horizon interval?

Thus, the problem really consists of the fact that the assumptions of the model imply that the simple linear relationships of equations (2.7), (3.8), and (3.23) hold (if at all) only for holding periods of a particular length, and we wish to be able to use the evaluation model based on these results to evaluate portfolio performance over all holding periods. We now intend to show that the linear relationships of equations (2.7), (3.8), and (3.23) hold for any arbitrary length of time as long as the returns R_F and R_M are expressed in terms of the "proper" compounding interval.

For the moment, let us assume that assets are priced *as if* the "true" horizon interval in the market were H-periods in length—where the dimension of a period is some small, arbitrary interval. We know then that if the capital asset pricing model is valid, the following holds for the jth security or portfolio:

$$E({}_HR_j) = {}_HR_F(1 - \beta_j) + E({}_HR_M)\beta_j\,, \tag{4.1}$$

where

$E({}_HR_j) = E(\Delta_H W_j/W_j) =$ the expected H-period rate of return for the jth security;

$E({}_HR_M)$ and ${}_HR_F$ are similar rates of return for the market portfolio M and the riskless security F; and

$\beta_j = \beta_{1j}$, β_{2j}, or β_{3j}, depending on the particular context in which we choose

[38] Formal definitions of these terms will be provided in Section V.

[39] The existence of what is here called the "horizon problem" became clear after several discussions with the members of the Finance Workshop. I am especially indebted to Professor Fisher for helping me to see the problem.

to interpret the concept of systematic risk.[40]

By equation (3.16) and the arguments given in note 31, we also know

$$ {}_HR_j \cong {}_HR_F(1 - \beta_j) + {}_HR_M\beta_j + e_j \,. \quad (4.2)$$

Now consider the rate of return ${}_NR_j$ on the jth security over an arbitrary N-period holding interval (where we assume N to be an integral multiple of H):

$$ \begin{aligned} {}_NR_j &= \left[\prod_{t=1}^{N/H} (1 + {}_HR_{jt})\right] - 1 \\ &= \left[\prod_{t=1}^{N/H} (1 + {}_HR_{Ft}(1 - \beta_j) \right. \\ &\qquad \left. + {}_HR_{Mt}\beta_j + e_{jt})\right] - 1 \,. \end{aligned} \quad (4.3)$$

As long as (1) R_{Ft} and $E(R_{Mt})$ are constant through successive H-period intervals, (2) the R_{Mt} and e_{jt} are distributed independently through time,[41] and (3) the e_{jt} and R_{Mt} are independent,[42] the expected N-period returns, $E({}_NR_j)$, are given by

$$ \begin{aligned} 1 + E({}_NR_j) &= [1 + E({}_HR_j)]^{1/\lambda} \\ &= [1 + (1 - \beta_j){}_HR_F \\ &\qquad + \beta_jE({}_HR_M)]^{1/\lambda} \,, \end{aligned} \quad (4.4)$$

where $\lambda = H/N$.

The reader will note that (4.4) is most certainly non-linear in β_j, since the expansion of the RHS will involve $\beta_j^{1/\lambda}$ and cross-product terms containing β_j. Hence, it is clear that the simple linear relationship of (4.1) will hold *only* for a holding interval which is H-periods in length. This is the essence of the "horizon problem."

Solving (4.4) for $E({}_HR_j)$, we have

$$ \begin{aligned} E({}_HR_j) &= [1 + E({}_NR_j)]^{\lambda} - 1 \\ &= (1 - \beta_j){}_HR_F + \beta_jE({}_HR_M) \,. \end{aligned} \quad (4.5)$$

But now what are ${}_HR_F$ and $E({}_HR_M)$ on the RHS of (4.5) in terms of observable N-period rates? Under the assumptions of constant expectations and independence through time, we have

$$ {}_HR_F = [1 + {}_NR_F]^{\lambda} - 1 \,, \quad (4.6)$$

$$ E({}_HR_M) = [1 + E({}_NR_M)]^{\lambda} - 1 \,. \quad (4.7)$$

Hence, rewriting (4.5) in terms of the potentially observable quantities given in (4.6) and (4.7), we have:

$$ \begin{aligned} [1 + E({}_NR_j)]^{\lambda} - 1 &= (1 - \beta_j) \\ &\times [(1 + {}_NR_F)^{\lambda} - 1] \\ &+ \beta_j\{[1 + E({}_NR_M)]^{\lambda} - 1\} \,. \end{aligned} \quad (4.8)$$

Now this relationship still holds if we divide both sides by λ:

$$ \begin{aligned} \frac{[1 + E({}_NR_j)]^{\lambda} - 1}{\lambda} &= (1 - \beta_j) \\ &\times \frac{[1 + {}_NR_F]^{\lambda} - 1}{\lambda} + \beta_j \\ &\times \frac{[1 + E({}_NR_M)]^{\lambda} - 1}{\lambda} \,. \end{aligned} \quad (4.9)$$

Define

$$ E(R_j^*) = \frac{[1 + E({}_NR_j)]^{\lambda} - 1}{\lambda} \quad (4.10)$$

and R_F^* and $E(R_M^*)$ likewise. The transformed rates, R^*, are just nominal N-period rates with H-period compounding intervals,[43] and in terms of this notation (4.9) becomes

$$ E(R_j^*) = (1 - \beta_j)R_F^* + \beta_jE(R_M^*) \,. \quad (4.11)$$

[40] Since the distinction between the three alternative interpretations of β_j are unimportant to the discussion here, we simply ignore the subscripts 1, 2, and 3 used in Section III.

[41] See Fama [12], Roll [46], and the papers reprinted in Cootner [9] for evidence on the serial independence of security returns.

[42] By the construction of R_M (eq. [3.5]) we know that cov $(R_{Mt}, e_{jt}) = x_j\sigma^2(e_j)$. But since x_j is on the average smaller than 1/1000, we ignore this covariance term for the sake of simplicity in deriving (4.4). Thus, there is a slight degree of approximation in equation (4.4).

[43] Note that

$$ \frac{[1 + E({}_NR_j)]^{\lambda} - 1}{\lambda} = \frac{N}{H} E({}_HR_j) \,, \quad (4.10a)$$

where, as before, $E({}_HR_j)$ is the expected rate of re-

Thus, equations (4.9) or (4.11) tell us that the simple linear relationship of (2.7), (3.8), and (3.23) will hold for returns calculated over a holding period of any length as long as we state the returns in terms of the "proper" compounding interval. But of course this result is empirically meaningless unless we can somehow determine the "proper" compounding interval, that is, unless we can determine H. We shall now turn our attention to this question.

B. SOME CONSIDERATIONS REGARDING THE "MARKET HORIZON INTERVAL"

There are several arguments which lead us to conclude that the market horizon interval is instantaneous. First, within the strict confines of the assumptions of our model regarding the perfect liquidity of all assets (i.e., transaction costs are zero), all investors will have instantaneous horizons[44] as long as portfolio evaluations are costless. Although these zero cost assumptions most certainly are not met in the real world, it may very well turn out that market prices behave *as though* they were fulfilled.

Second, an instantaneous "market horizon" would also be consistent with the assumption that an infinite number of investors all have non-zero horizon periods, but these horizon periods are distributed such that at every instant of time a large number of investors are making portfolio decisions and trading in the market. In addition, if we require the market to be in equilibrium at each instant, it must follow that the resulting "market horizon" is instantaneous. That is, prices behave *as though* all investors had instantaneous horizon periods.

On the basis of these arguments, let us consider the limit of (4.9) as the length of the horizon interval, H, goes to zero. Since, through the use of L'Hospital's rule,[45]

$$\lim_{\lambda \to 0} \frac{x^\lambda - 1}{\lambda} = \log_e x , \qquad (4.12)$$

we have

$$\begin{aligned}\lim_{H \to 0} \frac{[1 + E({}_NR_j)]^{H/N} - 1}{H/N} &= \log_e [1 + E({}_NR_j)] \\ &= (1 - \beta_j) \log_e (1 + {}_NR_F) + \beta_j \log_e [1 + E({}_NR_M)]\end{aligned} \qquad (4.13)$$

and hence for H/N close to zero

$$E(R_j^*) \cong \log_e [1 + E({}_NR_j)] . \qquad (4.14)$$

Thus, as long as the market horizon interval, H, is very small, we may use the log form (4.13) as a very good approximation to (4.9).[46]

---

turn on the jth security expressed in terms of an H-period compounding interval. Thus, the quantity $(N/N)E(HR_j)$ is just the expected *nominal N-period rate* of return on the jth security under the assumption of an H-period compounding interval.

[44] See Fama and Miller [20] for a discussion of this point.

[45] L'Hospital's rule:

$$\lim_{\lambda \to 0} \frac{x^\lambda - 1}{\lambda} = \lim_{\lambda \to 0} \frac{d(x^\lambda - 1)}{d\lambda} \Big/ \frac{d\lambda}{d\lambda}$$

$$= \lim_{\lambda \to 0} x^\lambda \log_e x = \log_e x .$$

[46] The reader is urged to remember that the logarithm is a very good *approximation* to the transformation given by (4.10), since a literal interpretation of the logarithmic form will lead to difficulty in interpreting arguments presented later.

We also note that $\log_e [1 + E({}_NR_j)]$ may be given an intuitive interpretation in the same manner as

$$E(R_j^*) = \frac{[1 + E({}_NR_j)]^{H/N} - 1}{H/N} .$$

Whereas $E(R_j^*)$ is the nominal *N-period* (return under the assumption of an H-period compounding interval (see n. 43 below), $\log_e(1 + {}_NR_j)$ is also the *N-period* rate of return but expressed in terms of continuous compounding or an infinitely small compounding interval.

An aggregation problem.—Consider now the nominal N-period expected returns $[E(R_p^*)]$ on a portfolio consisting of K securities where $E(R_p^*)$ is defined as in (4.10). Let y_j ($j = 1, 2, \ldots, K$) be the fraction of the portfolio invested in the jth security and let $f(R_j) = R_j^*$. Then, noting that

$$W_{PN}/W_{P0} = \sum_{j=1}^{K} y_j(W_{jN}/W_{j0})$$

and that

$$W_{jN}/W_{j0} = f^{-1}(R_j^*) = \left(\frac{H}{N}R_j^* + 1\right)^{N/H}$$

we have from (4.9) and (4.10) that

$$E(R_p^*) = f\Big[\sum_{j=1}^{K} y_j f^{-1}[E(R_j^*)]\Big]$$

$$= f\Big\{\sum_{j=1}^{K} y_j\Big[\frac{H}{N}[R_F^*(1 - \beta_j) + E(R_M^*)\beta_j] + 1\Big]^{N/H}\Big\}. \quad (4.15)$$

Since the expansion of the bracketed terms on the RHS of (4.15) will involve cross-product terms containing the β_j, it is clear that the systematic risk of the portfolio, β_p, is a function of the individual coefficients β_j, the riskless rate, and the expected return on the market portfolio. In fact, the risk of the portfolio, β_p, will be strictly stationary through time only if the assets of the portfolio are continuously redistributed to maintain the fractions y_j at their original values. Hence, one must be extremely careful about aggregating the risk coefficients of individual securities to obtain the risk of a portfolio. However, given β_p (which can be estimated for the portfolio as a whole), we may write

$$E(R_p^*) = (1 - \beta_p)R_F^* + \beta_p E(R_M^*)\,. \quad (4.16)$$

Thus, as (4.16) shows, the expected N-*period* returns on the portfolio can be expressed as a linear function of the risk-free rate and the expected return on the market portfolio as long as we express these N-period rates in terms of a compounding interval of H-periods. We shall now consider some arguments regarding the length of the "market-horizon interval," H.

C. IMPLICATIONS OF THE HORIZON SOLUTION FOR THE MEASURE OF SYSTEMATIC RISK

One of the most important implications of the horizon solution given above and the restatement of the capital asset pricing results in the form of (4.11) is the fact that the measure of systematic risk, β_j, may be used for a holding period of any length, N. That is, we shall now show that as long as H, the "market horizon," is instantaneous, the expected value of the estimate of systematic risk, $\hat{\beta}_j$, will be independent of the length of time (N) over which the sample returns are calculated[47] and will be equal to the true value β_j. This result, of course, implies that we can use a given measure of risk for a portfolio to evaluate the portfolio's performance over a horizon of *any* length. In addition, it means that we need only concern ourselves with the problem of obtaining the "best" estimate of β_j for any security or portfolio j, and any investor, regardless of his decision horizon, will be able to use that measure in arriving at an optimal portfolio decision.

The Gaussian case.—Consider first the definition of systematic risk in the context of Gaussian distributions. Recalling the specifications of the market model

[47] That is, given the total calendar time interval of observation (and ignoring the effects on sampling error for the moment), the solution implies the expected value of the estimate of β_j will be independent of the length of the subintervals over which the sample returns are calculated, be they daily, monthly, quarterly, etc.

given in (3.2) and (3.3), it was shown that

$$\beta_{2j} = \frac{\text{cov }(R_j,R_M)}{\sigma^2(R_M)} = \frac{b_j\sigma^2(\pi) + x_j\sigma^2(e_j)}{\sigma^2(R_M)}\,. \quad (3.9)$$

But since it was also shown in Section III that $\sigma^2(R_M) \cong \sigma^2(\pi)$, (3.9) reduces to

$$\beta_{2j} = \frac{\text{cov }(R_j,R_M)}{\sigma^2(R_M)} \cong b_j + \frac{x_j\sigma^2(e_j)}{\sigma^2(R_M)}\,, \quad (3.12)$$

which is the form used in deriving the measure of portfolio performance. We are concerned here with the estimate of β_{2j}, given by cov $(R_j, R_M)/\sigma^2(R_M)$. We emphasize that (3.9) and (3.12) were derived strictly within the confines of a single-period model within which the relevant covariances and variances refer to the properties of the set of probability distributions on one-period returns. Within this context, of course, there is no ambiguity regarding the interpretation of H, the market horizon; it is determined by the length of the period.

We now wish to consider the expected value of the following estimate, $\hat{\beta}_{2j}$, derived from a sample of N-period returns observed over time:[48]

$$E(\hat{\beta}_{2j}) = \frac{\text{cov }(R_j^*,R_M^*)}{\sigma^2(R_M^*)}\,. \quad (4.17)$$

First note that $1 + {}_NR_j$ (from which R_j^* is derived) is given by

$$1 + {}_NR_j = \prod_{k=1}^{N/H} (1 + {}_HR_{jk})$$

for integer N/H. Furthermore, we know that

$$\underset{H\to 0}{\text{limit}} = \prod_{k=1}^{N/H} (1 + {}_HR_{jk}) = \exp\left(\sum_{k=1}^{N/H} {}_HR_{jk}\right)\,. \quad (4.18)$$

Using (4.10), (4.12), (4.17), and (4.18), the assumptions of stationarity and serial independence, and taking the limit, we have

$$\underset{H\to 0}{\text{limit}}\, E(\hat{\beta}_{2j}) = \underset{H\to 0}{\text{limit}} \frac{\text{cov}\left\{\dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{jk}\right)\right]^{H/N} - 1}{H/N}, \dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{Mk}\right)\right]^{H/N} - 1}{H/N}\right\}}{\sigma^2\left\{\dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{MK}\right)\right]^{H/N} - 1}{H/N}\right\}}$$
$$= \frac{\text{cov}\left\{\underset{H\to 0}{\text{limit}}\dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{jk}\right)\right]^{H/N} - 1}{H/N}, \underset{H\to 0}{\text{limit}}\dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{Mk}\right)\right]^{H/N} - 1}{H/N}\right\}}{\sigma^2\left\{\underset{H\to 0}{\text{limit}}\dfrac{\left[\exp\left(\sum_{k=1}^{N/H} {}_HR_{Mk}\right)\right]^{H/N} - 1}{H/N}\right\}}$$
$$= \frac{\text{cov }({}_HR_j,{}_HR_M)}{\sigma^2({}_HR_M)}\,. \quad (4.19)$$

[48] We assume, of course, that the probability distributions generating the sample observations are stationary.

Thus, as long as the sample data are transformed according to (4.10), the expected value of the estimates of β_{2j} will be independent of the length of time over which the returns are calculated. If this result is empirically true, it is extremely important. Our earlier results imply that we can evaluate portfolios over a horizon of any length, even if different from the market horizon. The results of (4.19) imply that we may also estimate the systematic risk of the portfolio without regard for the particular horizon interval for which we intend to use it. Hence, we may calculate the measure of systematic risk on the basis of the most efficient sample available, whether it be daily, monthly, or yearly data,[49] and all investors, regardless of horizon length, can use it in evaluating and selecting portfolios.

The infinite variance case.—The problem of estimating β_{3j} in the context of non-Gaussian Stable distributions reduces to the same result as above, except that the economic interpretation is slightly different from that in the finite variance case. The difficulty arises, of course, because the covariance is not defined in this context. However, Wise [68] has shown (for the case of non-stochastic regressors) that as long as $\alpha > 1$, the least-squares estimates of b_j in the Stable market model are unbiased and consistent although not efficient. In addition, Monte Carlo evidence presented by Fama and Babiak [17] suggests that the use of least-squares procedures in Stable models like that of (3.2) is not completely inappropriate.[50] Thus, in light of this evidence, we define our estimate of b_j to be

$$b_j = \frac{\sum_{t=1}^{T}(R_{jt}^* - \bar{R}_j^*)(\pi_t^* - \bar{\pi}^*)}{\sum_{t=1}^{T}(\pi_t^* - \bar{\pi}^*)^2}, \quad (4.20)$$

where T is the total number of observations and the barred variables represent mean values. By the arguments given in the derivation of the Stable market model, $\beta_{3j} \cong b_j$. Thus, all we need now is a measure of the market factor π. But King [34, p. 190] found that explicit estimates of the market factor (obtained with factor analytic techniques) were correlated .97 with the Standard & Poor Index for the period 1927–60. As we shall see in Section VI, the Standard & Poor Index is the index which most closely meets the definition of M, the market portfolio, so on this basis we rewrite (4.20) as

$$b_j \cong \frac{\sum_{t=1}^{T}(R_{jt}^* - \bar{R}_j^*)(R_{Mt}^* - \bar{R}_M^*)}{\sum_{t=1}^{T}(R_{Mt}^* - \bar{R}_M^*)^2} \quad (4.21)$$
$$\cong \beta_{3j}\,.$$

D. THE MEASURE OF PORTFOLIO PERFORMANCE

The discussion above indicates that the measure of risk derived in the context of a single-period homogeneous horizon model will extend quite readily to a world in which trading takes place continuously and where investors have heterogeneous horizon periods. All we need do is restate (3.18) and (3.30) in

[49] Given the total calendar length of the sample interval, purely statistical considerations would indicate using the smallest observation interval possible in order to maximize the number of observations. However, gathering daily data will usually be far more expensive than gathering monthly or quarterly data, and one has to take these costs into consideration when deciding on the "optimal" sample size and interval of observation.

[50] There has been very little investigation into the properties of alternative estimators in stable models such as ours, and until additional insights are obtained, we are forced to proceed with least-squares procedures. However, recent work by Robert Blattberg and Thomas Sargent ("Regression with Paretian Disturbances: Some Sampling Results" [unpublished mimeographed paper, Carnegie-Mellon University, April, 1968]) indicates that least-squares procedures may be quite acceptable for small sample sizes.

terms of the transformed returns R^* given by (4.10). As shown in Section IV-C above, the estimating procedures for β_{2j} and β_{3j} are identical—the only difference being in the interpretation of the result. Thus, to simplify the exposition we shall henceforth couch the discussion in terms of β_j (without a subscript 2 or 3), where β_j may represent either β_{2j} or β_{3j} as the reader pleases. At any point where confusion may arise, we shall revert to the explicit notation.

By applying the arguments of Section III to equation (4.11), the revised measure of performance implied by the horizon solution is obtained as

$$\begin{aligned}\delta_j^* &= R_j^* - E(R_j^* \mid R_M^*, \beta_j) \\ &= R_j^* - [R_F^*(1 - \beta_j) + R_M^* \beta_j] \qquad (4.22) \\ &\cong e_j^* ,\end{aligned}$$

where δ_j^* is to be interpreted analogously to either δ_{2j} defined in (3.18) or as δ_{3j} defined in (3.30), depending on the interpretation of β_j. The variable e_j^* is analogous to the disturbance e_j defined in equations (3.2) and (3.3).

V. THE EVALUATION CRITERIA AND THE CONCEPT AND MEASUREMENT OF EFFICIENCY

A. THE EVALUATION CRITERIA

A measure of portfolio performance which provides a measure of a manager's ability to pick "winners" was developed in the preceding sections, culminating in the final form given by equation (4.22). The problem we address at this point is the determination of the criteria by which we judge the performance of any particular portfolio. In Part B of this section we shall derive a measure of a portfolio's "efficiency," and in Part C we shall discuss the relationship between the measures of efficiency and performance. Since all the assumptions made in Section III regarding the disturbance terms e_j also apply to e_j^*, we are led quite naturally to the following criteria for the evaluation of an estimate (or series of estimates) δ_{jt}^* for a particular portfolio over some time period t.

Criterion for "neutrality."—A portfolio's performance will be defined as neutral if its historical returns are equal to those which the capital asset pricing model implies it should have earned given its level of systematic risk. Formally, this means the results should meet the following conditions:

$$E(\delta_j^*) = E(e_j^*) = 0 , \qquad (5.1)$$

$$E(\delta_{j,t}^* \delta_{j,t+\tau}^*) = E(e_{j,t}^* e_{j,t+\tau}^*) = 0 \qquad \tau \neq 0 . \qquad (5.2)$$

That is, we expect the portfolio to experience returns through successive holding periods which will cause it to fluctuate randomly about the market line $R_F MQ$ portrayed in Figure 4.

Thus, a neutral portfolio is one on which the returns are no better or worse than those which could have been earned by a comparable naïve *FM* portfolio. A neutral portfolio may also be interpreted as one which does no better or worse than that which could have been achieved by a randomly selected portfolio with identical systematic risk.

Criterion for "superiority."—A superior portfolio will be defined as one which, through successive holding periods, realizes returns such that

$$E(\delta_j^*) > 0 . \qquad (5.3)$$

Thus, a superior portfolio is defined as a portfolio whose returns are consistently greater than those implied by its level of systematic risk. Hence, the returns on such a portfolio would be greater than those which could have been earned by a random selection buy-and-hold policy or by a naïve investment in an *FM* portfolio having identical systematic risk.

Recalling our earlier discussion in Section I regarding the martingale hypothesis, it is clear that (5.3) also defines the criterion for judging a portfolio manager to be a superior analyst. A portfolio manager who possesses superior economic insight and thus the ability (1) to forecast some of the factors affecting future disturbances (e_j^*) for particular securities or (2) to make better than average forecasts of the future realizations on the market factor π, will be able to create a portfolio which consistently dominates the market line R_FMQ of Figure 4. We might mention here that the existence of portfolios satisfying (5.3) is inconsistent with the strong form of the martingale hypothesis given by (1.2).

Criterion for "inferiority."—We define an inferior portfolio to be one which, through successive holding periods, realizes results such that it is consistently dominated[51] by the market line R_FMQ of Figure 4 and thus has

$$E(\delta_j^*) < 0 . \qquad (5.4)$$

The martingale property of security price movements implies that the best estimate of future prices (barring superior information) is merely the present price plus a normal expected return. Since any naïve investor or portfolio manager in the market could easily follow this forecasting procedure and expect, on the average, to do as well as the market as a whole, we conclude (if the strong form of the martingale hypothesis is correct) that an inferior portfolio can exist only because the portfolio managers pursue activities which generate expenses. These expenses must be paid out of income, and thus the portfolio returns are reduced.

It should be noted when evaluating mutual funds that there are expenses generated in the provision of services which benefit shareholders (the provision of bookkeeping services is an example), and the value of these benefits to shareholders should be taken into consideration. However, there may be other unnecessary expenses generated which cause the returns to be lower than expected. For example, there may very well be portfolio managers who pursue activities such as attempting to forecast security prices (and trading securities on the basis of these forecasts) while they are unable to increase returns enough to cover their research and commission expenses.

B. THE CONCEPT AND MEASUREMENT OF EFFICIENCY

The concept of efficiency.—The reader is cautioned to beware of confusing the above definitions of *performance* with the concept of *efficiency* in the Markowitz-Tobin-Sharpe sense. An efficient portfolio is one which provides maximum expected return for a given level of "risk" and minimum "risk" for a given level of of expected return. It is important to note here that "risk" in the definition of efficiency refers to the *total* risk of the portfolio and not just its *systematic* risk (which must always be less than or equal to a portfolio's total risk). Under the as-

[51] The exact meaning of "consistently dominated" is left undefined at this point and will be considered below in the context of the empirical results. It will suffice to say at this point that a portfolio can be above or below the efficient boundary either because of random factors or because the portfolio is systematically better or worse than the market portfolio. In addition, if one is examining many portfolios, it is reasonable to expect some of them to be consistently better or worse during the sampling period for purely random reasons. A detailed discussion of this point is contained in Jensen [32].

sumptions stated in Section II, it was shown that any efficient[52] portfolio ϵ will satisfy

$$E[R_\epsilon | E(R_M), \sigma(R_\epsilon)/\sigma(R_M)] = R_F + [E(R_M) - R_F] \frac{\sigma(R_\epsilon)}{\sigma(R_M)}, \quad (2.6a)$$

where $\sigma(R_\epsilon)/\sigma(R_M)$ is the total relative risk of the portfolio ϵ. Recall that the results of the capital asset pricing model (given in [2.7]) merely state the returns which should be expected on any asset given its level of systematic risk. We emphasize that if the capital asset pricing model is valid, (2.7) applies to *any* asset or portfolio. On the other hand, (2.6a) will be satisfied only by efficient portfolios as portrayed in Figure 2. The boundary of the opportunity set, the line R_FMQ in Figure 2, is given by equation (2.6a). The only portfolios satisfying the requirements for efficiency lie along this line, and (in the absence of superior information about future security returns) all other feasible portfolios lie to the right and below this line.

It should be noted that we are now abandoning the assumption of homogeneous expectations. The reader should now interpret the opportunity set portrayed in Figure 2 as the set which would be determined by knowledge of only the parameters of the market model for each security and the parameters of the distribution on the market factor[53] Any investor or portfolio manager in possession of information which enables him to (correctly) form expectations on π and e_j which are non-zero will be able to form portfolios which dominate the naïve no-superior-information opportunity set. We shall henceforth use the word *efficient* to refer to this "naïve" concept of efficiency and in particular will *not* use it to refer to the set of "dominant" portfolios which any *individual* investor might observe given any special information he might have regarding the future realizations of the market factor π and the disturbances e_j.

Within this context, then, (2.6a) gives us the expected returns on any efficient portfolio ϵ conditional on the expected returns on the market portfolio and the total relative risk of the portfolio. (The reader is reminded that it is implicitly assumed in this definition of efficiency that $E[\pi] = 0$ and $E[e_j] = 0$, $[j = 1, 2, \ldots, N]$.)

Let us now consider the derivation of an expression for the expected return on any *efficient* portfolio ϵ conditional on the *realized* returns on the market portfolio rather than the expected returns. Adding $\beta_\epsilon\pi + e_\epsilon$ to both sides of (2.6a), we have

$$E[R_\epsilon | E(R_M), \sigma(R_\epsilon)/\sigma(R_M)] + \beta_\epsilon\pi + e_\epsilon = R_F + [E(R_M) - R_F] \times \frac{\sigma(R_\epsilon)}{\sigma(R_M)} + \beta_\epsilon\pi + e_\epsilon, \quad (5.5)$$

and since for all *efficient* portfolios

$$E[R_\epsilon | E(R_M), \sigma(R_\epsilon)/\sigma(R_M)] = E(R_\epsilon), \quad (5.6)$$

we have, from (3.2) and the fact that $\beta_\epsilon \cong b_\epsilon$ (by the arguments given in Section III), that to a close approximation

$$E[R_\epsilon | E(R_M), \sigma(R_\epsilon)/\sigma(R_M)] + \beta_\epsilon\pi + e_\epsilon \cong R_\epsilon. \quad (5.7)$$

Using (5.7), we can write (5.5) as

$$R_\epsilon \cong R_F + [E(R_M) - R_F] \times \frac{\sigma(R_\epsilon)}{\sigma(R_M)} + \beta_\epsilon\pi + e_\epsilon. \quad (5.8)$$

[52] We shall assume throughout the following discussion that security returns are normally distributed. We shall deal with the case of non-Gaussian Stable distributions at the end of Section V.

[53] That is, we assume knowledge of only $E(R_j)$, β_j, $E(e_j) = 0$, $\sigma^2(e_j)$, $E(\pi) = 0$, and $\sigma^2(\pi)$.

But since, by (3.5) and the arguments of note 31 below,

$$\pi \cong R_M - E(R_M) \, , \qquad (5.9)$$

we can substitute into (5.8) and arrive at

$$R_\epsilon \cong R_F + [E(R_M) - R_F] \frac{\sigma(R_\epsilon)}{\sigma(R_M)} + [R_M - E(R_M)]\beta_\epsilon + e_\epsilon \, . \qquad (5.10)$$

Note that

$$\frac{\sigma(R_j)}{\sigma(R_M)} = \frac{1}{r_j}\beta_j \qquad r_j \neq 0 \, , \qquad (5.11)$$

where r_j is the product-moment correlation coefficient between the returns on the jth portfolio and the returns on the market portfolio.

Using (5.11), adding and subtracting $\beta_\epsilon R_F$ on the RHS of (5.10) and rearranging, we have for all efficient portfolios

$$R_\epsilon \cong R_F + (R_M - R_F)\beta_\epsilon + [E(R_M) - R_F]\beta_\epsilon\left(\frac{1}{r_\epsilon} - 1\right) + e_\epsilon \qquad (5.12)$$
$$r_\epsilon \neq 0 \, .$$

Now, since $E(e_j) = 0$ for all j by (3.3a), we have $E(e_\epsilon) = 0$ and

$$E[R_\epsilon | E(R_M), R_M, \beta_\epsilon, \sigma(R_\epsilon)/\sigma(R_M)] \cong R_F + (R_M - R_F)\beta_\epsilon + [E(R_M) - R_F]\beta_\epsilon\left(\frac{1}{r_\epsilon} - 1\right) \qquad (5.13)$$
$$r_\epsilon \neq 0 \, .$$

Equation (5.13) is an important result. It gives us the expected return on any efficient portfolio ϵ conditional on the realized returns on the market portfolio, its systematic risk, and its total relative risk. But note also that we are left with a term involving $E(R_M)$ which indicates that we cannot define efficiency without taking into account the ex ante expected returns on the market portfolio.

In considering this result, note that the first two terms on the RHS of (5.13) are identical to those in (3.17) used in the definition of "performance." These two terms tell us what the portfolio should earn given its level of systematic risk. However, if the portfolio is also to be efficient, its returns must be higher by an amount given by

$$[E(R_M) - R_F]\beta_\epsilon\left(\frac{1}{r_\epsilon} - 1\right) \qquad r_\epsilon \neq 0 \, . \qquad (5.14)$$

Let us define a perfectly diversified portfolio as one for which the total risk of the portfolio is equal to its systematic risk, and hence one for which $r_j = 1$. Now the quantity

$$\beta_j\left(\frac{1}{r_j} - 1\right) = \frac{\sigma(R_j)}{\sigma(R_M)} - \beta_j \qquad (5.15)$$

is just the increment in the portfolio's risk (measured, of course, in a relative sense) which is due to the lack of perfect diversification.

In the absence of transactions costs, a rational manager would never hold an imperfectly diversified portfolio[54] unless he believed he could forecast future security prices to some extent. If he believed he could forecast future prices successfully, it would most certainly be rational to sacrifice some diversification and concentrate some of the portfolio's holdings in those select securities with the highest expected "abnormal" return.[55] But to the extent that the manager accepts additional risk in acting on his forecasts, he must earn higher returns to compensate for it or the portfolio will be inefficient in the sense that a perfectly diversified *FM* portfolio with the same (higher) level of total risk would earn higher returns. By our previous arguments, $\beta_j[(1/r_j) - 1]$ represents the incremental risk due to the lack of

[54] That is, anything other than an *FM* portfolio defined in Section III.

[55] That is, those for which the manager believes $E(e_j) > 0$ are largest.

perfect diversification, and the term $[E(R_M) - R_F]$ in (5.14) is the expected premium per unit of risk. Thus, (5.14) represents the additional returns which must be earned by an imperfectly diversified portfolio in order for it to be efficient.

Before going on to define an explicit measure of efficiency, let us digress briefly to provide an intuitive interpretation of the foregoing concepts and issues. In considering the definition of a measure of efficiency, one is tempted to simply replace the term $E(R_M)$ in equation (2.6a) with R_M and interpret the resulting expression as one defining the expected returns on any efficient portfolio conditional on the realized returns on the market portfolio.[56] That is, it is tempting to simply relabel the vertical axis in Figure 2 as R instead of $E(R)$ and to interpret the line R_FMQ as representing the locus of points about which all efficient portfolios will scatter. It is clear from equation (5.12) that the realized returns on all efficient portfolios *will not* scatter about such a simple straight line in the ex post return and risk plane, since the *expected* returns on the market portfolio also appear in the equation. To see the issues more clearly, consider the three situations portrayed in panels A, B, and C of Figure 5, in which the ex post returns of a hypothetical portfolio k are plotted against its systematic risk, β_k, and total relative risk, $\sigma(R_k)/\sigma(R_M)$. The panels differ only in the assumed values of the realized returns on the market portfolio. In panel A it is assumed that $R_M = E(R_M)$, in panel B it is assumed that $R_M < E(R_M)$, and in panel C, that $R_M > E(R_M)$.

Let us now consider our hypothetical portfolio k with a level of systematic risk of .5 and managed by an individual who attempts to forecast the future prices of individual securities. In attempting to incorporate his forecasts into the portfolio, the manager is forced to accept additional (and diversifiable) risk in the portfolio. We assume for illustrative purposes that this results in a total relative risk of $.75 = \sigma(R_k)/\sigma(R_M)$. Now equation (5.13) (with [5.15]) indicates that in order for this portfolio to be efficient the manager's forecasting efforts must increase the expected returns on the portfolio by an amount equal to $[E(R_M) - R_F]$ (.25), which is simply the amount of incremental (and diversifiable) risk in the portfolio multiplied by the ex ante price per unit of risk.

For illustrative purposes, let us also assume that our hypothetical manager actually cannot forecast any better than a random selection policy, and thus he is reimbursed in the market only for the amount of systematic risk he has taken (in this case, .5). Let us also assume (without loss of generality) that the error terms, e, are zero for all portfolios we shall consider in our example.

Now the points labeled a in panels A, B, and C denote the ex post returns, R_k, and *systematic* risk (.5) of our portfolio under the three different assumptions regarding the value of the realized return on the market portfolio M. The points labeled b in the figures denote the ex post returns and *total relative* risk (.75) of the portfolio k. The points labeled c in the figures denote the ex post returns, R_I, earned by all *imperfectly diversified efficient* portfolios with a total relative risk of .75 and a systematic risk of 0.50. Finally, the points labeled d in the figures denote the ex post return, R_p, of all *perfectly diversified efficient* portfolios with a total relative (and systematic) risk of 0.75.

In panel A, where we have assumed $R_M = E(R_M)$, there is no difficulty at all

[56] Indeed something similar to this is suggested or implied in references 3, 7, 8, 27, 28, 54, 55, 58, and 66.

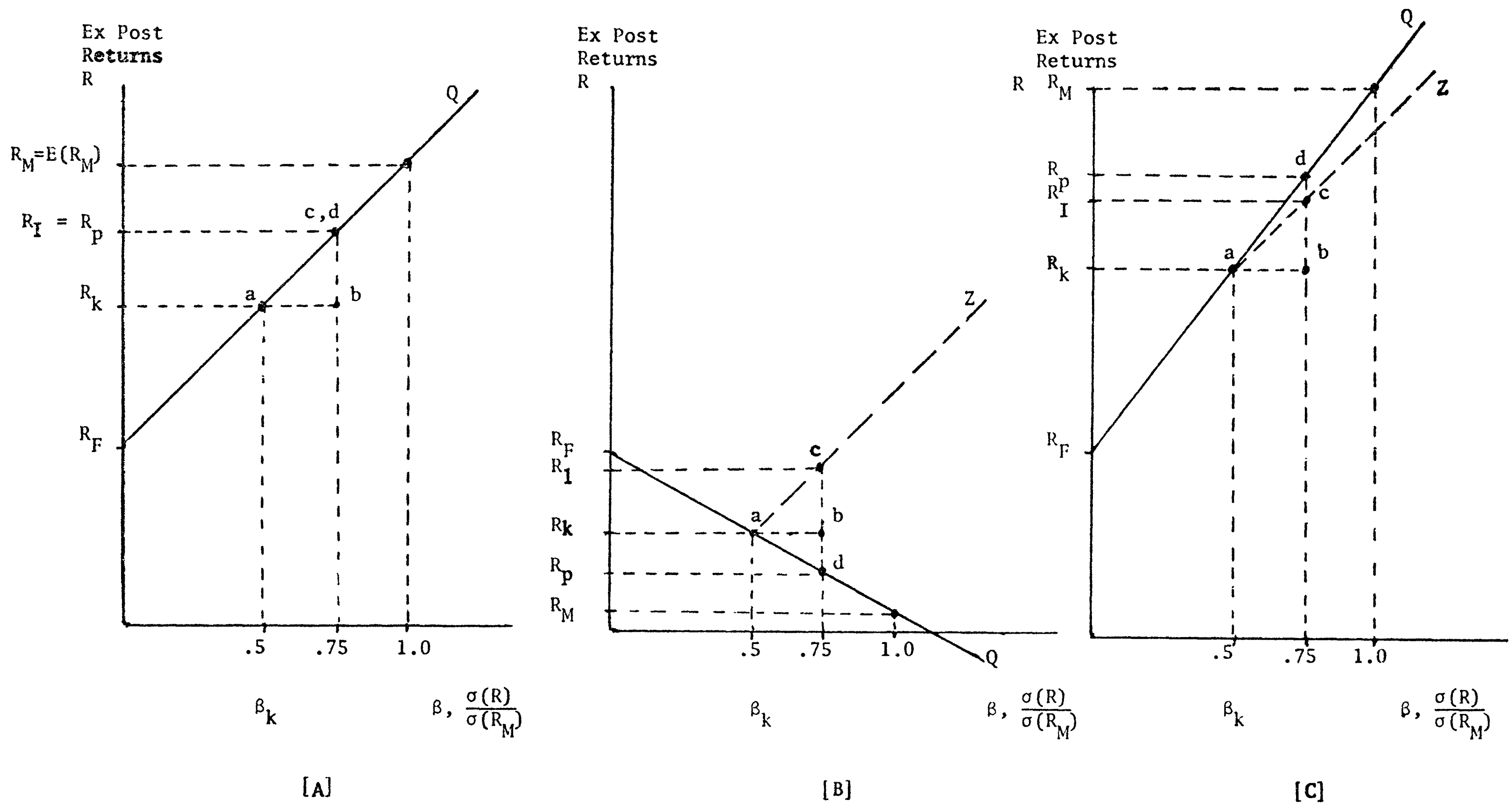

FIG. 5.—The relationship between the ex post returns earned by a perfectly diversified efficient portfolio p, an imperfectly diversified efficient portfolio I, and a hypothetical portfolio k, under three different assumptions regarding the ex post returns on the market portfolio.

in interpreting the diagram. All efficient portfolios (whether perfectly diversified or not) will scatter along the line R_FQ when their returns are plotted against their total relative risk. It is clear that our portfolio k (which by assumption is inefficient) at point b appears to be inefficient, since it is dominated by point c, d.

However, in panel B, where we have assumed $R_m < E(R_m)$, a simple interpretation of the "opportunity set" given by the solid line R_FQ is not valid. That is, all efficient portfolios will not lie along this line; only those portfolios which are perfectly diversified (i.e., for which $\beta_j = \sigma[R_j]/\sigma[R_M]$) will lie along this line. The point b in panel B again denotes the ex post returns and total relative risk of our hypothetical portfolio. But, contrary to the situation in panel A, point b appears to dominate point d, which represents the ex post return and risk of a perfectly diversified efficient portfolio. This impression is misleading. Point b looks better than point d only because the realized returns on the market portfolio were below the risk-free rate. The realized returns on the manager's imperfectly diversified portfolio were higher than those on the perfectly diversified portfolio p, because $\beta_k = .5$ while $\beta_p = .75$. It is clear that portfolio k cannot be efficient, since it is dominated by a perfectly diversified portfolio with identical returns and total relative risk of 0.5. Therefore, regardless of the realized returns on the market portfolio, if the imperfectly diversified portfolio k is to be efficient, the manager's forecasting ability must be good enough to reimburse the holders of the portfolio for the additional diversifiable risk taken. This increment in return is precisely the quantity given by equation (5.14), and the dashed line acZ in panel B denotes the ex post returns which must be earned by any imperfectly diversified portfolio with systematic risk equal to .5 in order for it to be efficient. The slope of acZ, of course, is determined by the ex ante risk premium per unit of risk. The difference between .75 and .5 is the incremental risk, and the difference between R_I and R_k is the incremental return necessary to compensate for this risk. (The reader will note that the line acZ is just one of an entire family of such lines emanating from every point on the line R_FQ.)

The case in which $R_m > E(R_m)$ is portrayed in panel C. Again, the level of systematic risk and the opportunity set R_FQ determine the ex post return R_k on our portfolio, and the ex ante risk premium determines the slope of the line acZ. The point c represents the point at which an imperfectly diversified efficient portfolio would lie with returns R_I.

The reader will note that if our hypothetical portfolio with a total relative risk of .75 were perfectly diversified (i.e., $\beta_k = .75$ also), and therefore efficient, it would have earned returns $R_p < R_k$ in the situation portrayed in panel B. Therefore, one might be tempted to conclude that the investor was actually better off with the imperfectly diversified and inefficient portfolio with returns R_p. In a sense this is true, but one must be very careful about giving the manager credit for this situation, which must be due solely to good luck. That is, if he were forecasting R_M to be less than R_F, he would certainly have been far better off to hold only the riskless asset rather than hold an imperfectly diversified portfolio. In addition, as previously mentioned, it is misleading to compare the imperfectly and perfectly diversified portfolios along the return dimension. It is clear that the holder of the imperfectly diversified portfolio k could have earned the same returns R_k with a perfectly diversified portfolio with total relative risk of .5 (rather than .75). Thus, the investor gained nothing from accepting this needlessly higher level of risk.

Moreover, if we consider the point d in panel C, which represents the perfectly diversified efficient portfolio with $\beta_p = \sigma(R_p)/\sigma(R_m) = .75$, it is clear that the returns R_p on such a portfolio are greater than R_k. Hence, in this case, the investor is not better off for having accepted the higher diversifiable risk for which he is not compensated.

In addition, the reader should note that in all three panels, both points c and d represent the locations of *efficient* portfolios with the same degree of total relative risk. Their returns will be coincident *only* when $R_m = E(R_m)$, and the differences are due solely to the random and unpredictable factors determining the returns on the market portfolio. The important point is that both portfolios I and p are ex ante efficient by definition; yet they may have vastly different ex post returns depending on whether $R_M \gtreqless E(R_M)$.

Now we shall consider the definition of a measure of efficiency and the evaluation criteria to be applied to it, and in Part C of this section we shall consider the relationship between the measures of performance and efficiency.

A measure of efficiency.—Utilizing (5.13) and taking account of the horizon solution, let us define a measure of efficiency, γ_j^*, as

$$\begin{aligned}\gamma_j^* &= R_j^* - E[R_j^* | E(R_M^*), R_M^*, \beta_j, \sigma(R_j^*)/\sigma(R_M^*)] \\ &= R_j^* - \left\{R_F + [R_M^* - R_F^*]\beta_j + [E(R_M^*) - R_F^*]\beta_j\left(\frac{1}{r_j} - 1\right)\right\} \qquad r_j \neq 0 \quad (5.16) \\ &\cong e_j^*,\end{aligned}$$

where, as before, e_j^* is defined analogously to equations (3.2) and (3.3). We consider now the criteria for judging a portfolio to be *"efficient," "inefficient,"* or *"superefficient."*

Criterion for "efficiency."—The arguments above imply that an efficient portfolio can be defined as one which through successive holding periods realizes returns such that[57]

$$E(\gamma_j^*) = E(e_j^*) = 0 . \qquad (5.17)$$

For the moment we shall ignore the problems associated with obtaining empirical estimates of $E(R_M^*)$, which of course are necessary for the estimation of γ^* We shall consider this point below.

Criterion for "inefficiency."—An inefficient portfolio will be defined as one for which

$$E(\gamma_j^*) < 0 . \qquad (5.19)$$

As noted above, it is perfectly possible that a manager is able to forecast security prices to some extent and still manage to create an inefficient portfolio. That is, it is possible that he might not earn returns sufficiently higher than a buy-and-hold policy to adequately compensate the holder of the portfolio for the additional risk taken due to the lack of perfect diversification.

Criterion for "superefficiency."—A portfolio will be defined to be superefficient if

$$E(\gamma_j^*) > 0 . \qquad (5.20)$$

One may question the possible existence of a superefficient portfolio, since we usually think of the efficient set of portfolios as dominating the set of feasible portfolios. However, recall that earlier we defined *efficiency* in terms of the op-

[57] Note also that since $\gamma_j^* \cong e^*$ we know also that

$$E(\gamma_{j,t}^*\gamma_{j,t+\tau}^*) = E(e_{j,t}^*e_{j,t+\tau}^*) = 0 . \quad (5.18)$$

portunity set which would be determined by knowledge of just the parameters of the market model for each security and the parameters of the distribution on the market factor. Hence, it is certainly possible for a manager with superior information or insight to create portfolios which dominate this "naïve" opportunity set.

C. THE RELATIONSHIP BETWEEN THE MEASURES OF EFFICIENCY AND PERFORMANCE

The case of perfectly diversified portfolios.—The concept of efficiency is extremely important, and it behooves us to investigate its relationship to the measure of portfolio performance, δ^*, suggested above. We have seen that a portfolio may be classified as inferior, neutral, or superior, and its classification depends on the manager's forecasting ability and the amount of expenses generated in the management of the portfolio. If a portfolio is either inferior or neutral, we can make unambiguous inferences regarding its efficiency. From the definition of the measure of performance, δ^*, given by (4.22), and the definition of the measure of efficiency, γ^*, given by (5.16), we see that

$$\gamma_j = \delta_j^* - [E(R_M^*) - R_F^*]\beta_j \times \left(\frac{1}{r_j} - 1\right) \qquad r_j \neq 0 . \tag{5.21}$$

The second term on the RHS of (5.21) is just the adjustment for the diversifiable risk in the portfolio and must be taken into account in measuring efficiency. Consider for the moment the case of a perfectly diversified portfolio. Since for such a portfolio $r_j = 1$, we know the last term on the RHS of (5.21) is zero. Thus, for a perfectly diversified portfolio, $\gamma_j^* = \delta_j^*$, and the measure of performance is also a measure of efficiency.[58]

The case of inferior portfolios.—If a portfolio is inferior, then it must also be inefficient. That is, if $E(\delta_j^*) < 0$, then $E(\gamma_j^*) < 0$ also. That is, by (5.21) and the fact that the last term on the RHS of (5.21) must always be positive,[59] we know that $\gamma_j^* \leq \delta_j^*$ always.

The case of superior portfolios.—The only case in which some ambiguity exists between the measure of performance, δ^*, and the inference regarding the efficiency of a portfolio is in the case of a superior forecaster with $E(\delta^*) > 0$. We can see from (5.21) and the definition of efficiency (5.17) that the superior portfolio will also be an efficient portfolio if

$$E(\delta_j^*) = [E(R_M^*) - R_F^*]\beta_j\left(\frac{1}{r_j} - 1\right) \qquad r_j \neq 0 . \tag{5.22}$$

That is, if the positive benefits of the forecaster's ability are just large enough to offset the effects of any imperfect diversification (represented by the difference between r_j and unity), the portfolio will be efficient.

In the situation where

$$E(\delta_j^*) > [E(R_M^*) - R_F^*]\beta_j\left(\frac{1}{r_j} - 1\right) \qquad r_j \neq 0 , \tag{5.23}$$

we define the portfolio to be superefficient, since the benefits from the superior forecasting ability are more than enough to offset the effects of the imperfect diversification.

Finally, in the situation where

$$0 < E(\delta_j^*) < [E(R_M^*) - R_F^*]\beta_j \times \left(\frac{1}{r_j} - 1\right) \qquad r_j \neq 0 , \tag{5.24}$$

[58] One might wonder at first whether a perfectly diversified portfolio can possibly be inefficient. The answer to such a question is yes, since all a manager of a perfectly diversified portfolio need do to make it inefficient is to generate expenses and therefore lower its returns.

[59] Since, under the assumption of risk aversion, $E(R_M)$ must be greater than R_F or no one would hold risky assets and, as an empirical fact, $\beta_j \geq 0$ and $r_j \geq 0$ always (cf. Blume [4] and Fama *et al.* [19]).

the portfolio is inefficient, since the benefits from the superior forecasting ability are not large enough to offset the effects of imperfect diversification.

It should be noted that, while a portfolio satisfying (5.24) is inefficient in and of itself, it most surely is a desirable investment if treated as a single asset in the context of an efficiently diversified portfolio. That is, the investor who realizes that $E(\delta_j^*) > 0$ may combine an investment in that portfolio with investments in other assets and hence create a portfolio which is in a sense superefficient. In effect, as soon as an investor realizes the superiority of a manager's forecasting ability, he may treat that ability as an additional asset in the opportunity set and thereby enable the efficient set (as viewed by himself) to shift upward and to the left.[60]

It is also interesting to note that this discussion regarding efficiency implies an economic justification for two very different types of funds: (1) funds which concentrate on maintaining perfectly diversified efficient portfolios and (2) special purpose funds that concentrate on being superior forecasters and perhaps ignore the diversification function entirely. Of course, the investor must realize these differences and treat them accordingly in building his own personal portfolio. The perfectly diversified efficient fund (with the proper risk level) is an appropriate investment for the investor's entire wealth stock. On the other hand, the special purpose fund need not be perfectly diversified (and in general cannot be) and may not be efficient as well, so that while it is a desirable asset to be included in the investor's total portfolio, it is not an appropriate investment for his entire wealth stock. (Of course, there is little if any justification for the existence of special purpose funds in the absence of superior forecasting ability.)

The concept of efficiency in the context of non-Gaussian Stable distributions.—Fama [16] has shown (using arguments analogous to those of Section II) that, in the context of non-Gaussian finite mean symmetric Stable distributions, an efficient portfolio ϵ must satisfy

$$E\left[R_\epsilon^* \middle| E(R_M^*), \frac{\gamma^{1/\alpha}(R_\epsilon^*)}{\gamma^{1/\alpha}(R_M^*)}\right] = R_F^* + [E(R_M^*) - R_F^*]\frac{\gamma^{1/\alpha}(R_\epsilon^*)}{\gamma^{1/\alpha}(R_M^*)}, \tag{5.25}$$

where γ is the dispersion parameter defined in Section III-C. Furthermore, by arguments analogous to those given in Section V-B, we can put (5.25) in terms of the ex post returns R_ϵ^* and R_M^*:

$$\begin{aligned} R_\epsilon^* &= E\left[R_\epsilon^* \middle| R_M^*, E(R_M^*), \frac{\gamma^{1/\alpha}(R_\epsilon^*)}{\gamma^{1/\alpha}(R_M^*)}\right] + e_\epsilon^* \\ &\cong R_F^* + [E(R_M^*) - R_F^*]\frac{\gamma^{1/\alpha}(R_\epsilon^*)}{\gamma^{1/\alpha}(R_M^*)} + [R_M^* - E(R_M^*)]\beta_\epsilon + e_\epsilon^* . \end{aligned} \tag{5.26}$$

[60] Of course, there is some question as to why a manager with such superior ability would sell his talents for anything less than their full value. This would imply that none of the benefits would be passed on to the fund investor and raises serious doubts that a superior mutual fund portfolio will ever be found. However, if the superior manager were a risk averter, he might find it advantageous to sell his talents for something less than their full expected value in return for a more stable income flow.

Equation (5.26) is analogous to (5.10), except that all the random variables are Stable variates fulfilling the assumptions stated in Section III-C and $\beta_\epsilon = \beta_{2\epsilon} \cong b_\epsilon$.

Unfortunately, there is no simple relationship between $\gamma^{1/\alpha}(R_\epsilon^*)/\gamma^{1/\alpha}(R_M^*)$ and β_ϵ comparable to (5.11). The lack of such a relationship prevents further simplification of (5.26) to a form like that of (5.13). From (3.21), (3.26), and the fact that[61] $\beta_j \cong b_j$, we know that

$$\frac{\gamma^{1/\alpha}(R_j^*)}{\gamma^{1/\alpha}(R_M^*)} \cong \frac{[\gamma(R_M)|\beta_j|^\alpha + \gamma(e_j)]^{1/\alpha}}{\gamma^{1/\alpha}(R_M)}, \tag{5.27}$$

and it is clear that

$$\frac{\gamma^{1/\alpha}(R_j^*)}{\gamma^{1/\alpha}(R_M^*)} \cong \beta_j \tag{5.28}$$

only when $\gamma(e_j) = 0$. Thus, given Stable distributions, a perfectly diversified portfolio ϵ must have $e_\epsilon^* = 0$ always, and this is equivalent to $r_\epsilon = 1$, given Gaussian distributions.

We can also see from (5.27) that just as in the case of Gaussian distributions, the total relative risk of an imperfectly diversified portfolio will always be greater than its systematic risk.

These arguments imply (1) that δ^*, the measure of performance, is also a measure of efficiency for all perfectly diversified portfolios, (2) that if $E(\delta^*) < 0$, then $E(\gamma^*) < 0$, that is, if the portfolio is inferior it must also be inefficient, and (3) that if the portfolio is imperfectly diversified (i.e., $\gamma^{1/\alpha}(R_j^*)/\gamma^{1/\alpha}(R_M^*) > \beta_j$), then $E(\delta_j^*) > 0$ must hold in order for the portfolio to be efficient.

[61] See pp. 184–85.

VI. AN APPLICATION OF THE MODEL TO THE EVALUATION OF MUTUAL FUND PORTFOLIOS

A. THE EMPIRICAL ESTIMATION OF THE MARKET MODEL AND SYSTEMATIC RISK

The data.—The sample consists of the portfolios of the 115 open-end mutual funds listed in Table 1. The funds included were all those for which net asset and dividend information was available in Wiesenberger's *Investment Companies* [67] for the ten-year period 1955–64.[62] Annual data were gathered for the period 1955–64 for all 115 funds, and as many additional annual observations as possible were collected for these funds in the period 1945–54.[63] For this earlier period, ten years of complete data were obtained for fifty-six of the original 115 funds.

Definitions of the variables.—The following are the exact definitions of the

[62] The data were gathered primarily from the 1955 and 1965 editions of Wiesenberger [67], but some data not available in these editions were taken from the 1949–54 editions. Data on the College Retirement Equities Fund (not listed in Wiesenberger) were obtained directly from annual reports. The last three digits of the identification numbers assigned to the funds correspond to the number of the page on which the fund is listed in the 1965 edition of Wiesenberger [67]. The College Retirement Equities Fund was arbitrarily assigned the number 1000. All per share data were adjusted for stock splits and stock dividends to represent an equivalent share as of the end of 1964.

[63] The reader is cautioned to remember, in interpreting the empirical results to follow, that these 115 funds do not actually represent 115 independent observations. That is, a mutual fund group composed of a number of separate funds with differing objectives (i.e., growth, income, and balanced) are often under identical management. In these cases, the fund strategies may very well not be independent. For instance, it is not uncommon to find the common stock portion of a balanced fund almost identical to the portfolio of a stock fund run by the same manager. In this event, we certainly do not have two independent observations. In addition, there is some indication that the fund groups do not choose strategies independently of one another; that is, there may be some funds which in essence "follow the leader."

TABLE 1

LISTING OF 115 OPEN-END MUTUAL FUNDS IN THE SAMPLE

ID NUMBER	CODE[1]	FUND
140	0	ABERDEEN FUND
141	0	AFFILIATED FUND, INC.
142	2	AMERICAN BUSINESS SHARES, INC.
144	3	AMERICAN MUTUAL FUND, INC.
145	4	ASSOCIATED FUND TRUST
146	0	ATOMICS, PHYSICS + SCIENCE FUND, INC.
147	2	AXE - HOUGHTON FUND B, INC.
1148	2	AXE - HOUGHTON FUND A, INC.
2148	0	AXE - HOUGHTON STOCK FUND, INC.
150	3	BLUE RIDGE MUTUAL FUND, INC.
151	2	BOSTON FUND, INC.
152	4	BROAD STREET INVESTING CORP.
153	3	BULLOCK FUND, LTD.
155	0	CANADIAN FUND, INC.
157	0	CENTURY SHARES TRUST
158	0	THE CHANNING GROWTH FUND
1159	0	CHANNING INCOME FUND, INC.
2159	3	CHANNING BALANCED FUND
160	3	CHANNING COMMON STOCK FUND
162	0	CHEMICAL FUND, INC.
163	4	THE COLONIAL FUND, INC.
164	0	COLONIAL GROWTH + ENERGY SHARES, INC.
165	2	COMMONWEALTH FUND - PLAN C
166	2	COMMONWEALTH INVESTMENT CO.
167	3	COMMONWEALTH STOCK FUND
168	2	COMPOSITE FUND, INC.
169	4	CORPORATE LEADERS TRUST FUND CERTIFICATES, SERIES 'B'
171	3	DELAWARE FUND, INC.
172	0	DE VEGH MUTUAL FUND, INC. (NO LOAD)
173	0	DIVERSIFIED GROWTH STOCK FUND, INC.
174	2	DIVERSIFIED INVESTMENT FUND, INC.
175	4	DIVIDEND SHARES, INC.
176	0	DREYFUS FUND INC.
177	2	EATON + HOWARD BALANCED FUND
178	3	EATON + HOWARD STOCK FUND
180	3	EQUITY FUND, INC.
182	3	FIDELITY FUND, INC.
184	3	FINANCIAL INDUSTRIAL FUND, INC.
185	3	FOUNDERS MUTUAL FUND
1186	0	FRANKLIN CUSTODIAN FUNDS, INC. - UTILITIES SERIES
2186	0	FRANKLIN CUSTODIAL FUNDS, INC. - COMMON STOCK SERIES
187	3	FUNDAMENTAL INVESTORS, INC.
188	2	GENERAL INVESTORS TRUST
189	0	GROWTH INDUSTRY SHARES, INC.
190	4	GROUP SECURITIES - COMMON STOCK FUND
1191	0	GROUP SECURITIES - AEROSPACE - SCIENCE FUND
2191	2	GROUP SECURITIES - FULLY ADMINISTERED FUND
192	3	GUARDIAN MUTUAL FUND, INC. (NO LOAD)
193	3	HAMILTON FUNDS, INC.
194	0	IMPERIAL CAPITAL FUND, INC.
195	2	INCOME FOUNDATION FUND, INC.
197	1	INCORPORATED INCOME FUND
198	3	INCORPORATED INVESTORS
200	3	THE INVESTMENT COMPANY OF AMERICA
201	2	THE INVESTORS MUTUAL, INC.
202	3	INVESTORS STOCK FUND, INC.
203	1	INVESTORS SELECTIVE FUND, INC.
205	3	INVESTMENT TRUST OF BOSTON

TABLE 1—*Continued*

206	2	ISTEL FUND, INC.
207	3	THE JOHNSTON MUTUAL FUND INC. (NO-LOAD)
208	3	KEYSTONE HIGH-GRADE COMMON STOCK FUND (S-1)
1209	4	KEYSTONE INCOME COMMON STOCK FUND (S-2)
2209	0	KEYSTONE GROWTH COMMON STOCK FUND (S-3)
210	0	KEYSTONE LOWER-PRICED COMMON STOCK FUND (S-4)
1211	1	KEYSTONE INCOME FUND-(K-1)
2211	0	KEYSTONE GROWTH FUND (K-2)
1212	1	THE KEYSTONE BOND FUND (B-3)
2212	1	THE KEYSTONE BOND FUND (B-4)
215	2	LOOMIS - SAYLES MUTUAL FUND, INC. (NO LOAD)
216	0	MASSACHUSETTS INVESTORS GROWTH STOCK FUND, INC.
217	3	MASSACHUSETTS INVESTORS TRUST
218	2	MASSACHUSETTS LIFE FUND
219	4	MUTUAL INVESTING FOUNDATION, MIF FUND
220	2	MUTUAL INVESTMENT FUND, INC.
221	0	NATIONAL INVESTORS CORPORATION
222	4	NATIONAL SECURITIES STOCK SERIES
1223	0	NATIONAL SECURITIES - GROWTH STOCK SERIES
2223	1	NATIONAL SECURITIES - INCOME SERIES
224	1	NATIONAL SECURITIES - DIVIDEND SERIES
225	2	NATION-WIDE SECURITIES COMPANY, INC.
226	2	NEW ENGLAND FUND
227	4	NORTHEAST INVESTORS TRUST (NO LOAD)
231	3	PHILADELPHIA FUND, INC.
232	4	PINE STREET FUND, INC. (NO LOAD)
233	3	PIONEER FUND, INC.
234	0	T. ROWE PRICE GROWTH STOCK FUND, INC. (NO LOAD)
235	1	PURITAN FUND, INC.
236	2	THE GEORGE PUTNAM FUND OF BOSTON
239	2	RESEARCH INVESTING CORP.
240	2	SCUDDER, STEVENS + CLARK BALANCED FUND, INC. (NO LOAD)
241	3	SCUDDER, STEVENS + CLARK COMMON STOCK FUND, INC. (NO LOAD)
243	3	SELECTED AMERICAN SHARES, INC.
244	2	SHAREHOLDERS' TRUST OF BOSTON
245	3	STATE STREET INVESTMENT CORPORATION (NO LOAD)
246	2	STEIN ROE + FARNHAM BALANCED FUND, INC. (NO LOAD)
247	0	STEIN ROE + FARNHAM INTERNATIONAL FUND, INC. (NO LOAD)
249	0	TELEVISION-ELECTRONICS FUND, INC.
250	0	TEXAS FUND, INC.
251	3	UNITED ACCUMULATIVE FUND
252	4	UNITED INCOME FUND
253	0	UNITED SCIENCE FUND
254	1	THE VALUE LINE INCOME FUND, INC.
255	0	THE VALUE LINE FUND, INC.
256	4	WASHINGTON MUTUAL INVESTORS FUND, INC.
257	2	WELLINGTON FUND, INC.
259	3	WISCONSIN FUND, INC.
260	2	COMPOSITE BOND AND STOCK FUND, INC.
1261	3	CROWN WESTERN-DIVERSIFIED FUND (D-2)
2261	2	DODGE + COX BALANCED FUND (NO LOAD)
2262	2	FIDUCIARY MUTUAL INVESTING COMPANY, INC.
263	4	THE KNICKERBOCKER FUND
267	4	SOUTHWESTERN INVESTORS, INC.
1268	2	WALL STREET INVESTING CORPORATION
2268	2	WHITEHALL FUND, INC.
1000	0	COLLEGE RETIREMENT EQUITIES FUND

[1] Wiesenberger classification as to fund investment objectives: 0 = growth, 1 = income, 2 = balanced, 3 = growth income and 4 = income growth.

variables used in the estimation procedures:

S_t = level of the Standard & Poor Composite 500 Price Index[64] at the end of year t.

D_t = estimate of dividends received on the market portfolio in year t as measured by annual observations on the four-quarter moving average[65] of the dividends paid by the companies in the Composite 500 Index (stated on the same scale as the level of the Standard & Poor 500 Index).

$R^*_{Mt} = \log_e\left(\frac{S_t + D_t}{S_{t-1}}\right)$ = the estimated annual continuously compounded rate of return on the market portfolio[66] M for year t.

NA_{jt} = per share net asset value of the jth fund at the end of year t.

ID_{jt} = per share "income" dividends paid by the jth fund during year t.

CG_{jt} = per share "capital gains" distributions paid by the jth fund during year t.

$R^*_{jt} = \log_e\left(\frac{NA_{jt} + ID_{jt} + CG_{jt}}{NA_{j,t-1}}\right)$ = the annual continuously compounded rate of return on the jth fund during year t (adjusted for splits and stock dividends).[67]

n_j = the number of yearly observations of the jth fund; $10 \leq n_j \leq 20$.

The empirical estimates.—It was shown in Section IV-C that we can use the same estimator for β_j regardless of whether we assume Gaussian distributions or symmetric non-Gaussian finite mean Stable distributions. Keeping these alternative interpretations in mind, let us define $\hat{\beta}_j$, the estimate of systematic risk for the jth portfolio obtained from annual data, as

$$\hat{\beta}_j = \sum_{t=1}^{n_j}(R^*_{jt} - \bar{R}^*_j)(R^*_{Mt} - \bar{R}^*_{Mj}) \Big/ \sum_{t=1}^{n_j}(R^*_{Mt} - \bar{R}^*_{Mj})^2 \qquad (6.1)$$

$$j = 1, 2, \ldots, 115,$$

where

$$\hat{\beta}_j = \begin{cases} \hat{\beta}_{2j} \text{ under the assumption of finite variances } (\alpha = 2) \\ \hat{\beta}_{3j} \text{ under the assumption of infinite variance symmetric Stable distributions with } 1 < \alpha < 2, \end{cases}$$

$$\bar{R}^*_j = \sum_{t=1}^{n_j} R^*_{jt} \Big/ n_j ,$$

and

$$\bar{R}^*_{Mj} = \sum_{t=1}^{n_j} R^*_{Mt} \Big/ n_j .$$

[64] Obtained from Standard & Poor [60].

[65] Obtained from Standard & Poor [60]. Since the use of this moving average introduces measurement errors in the index returns, it would be preferable to use an index of the actual dividends, but such an index is not available.

[66] As the capital asset pricing model implies, the market portfolio M is conceptually well defined as a portfolio consisting of an investment in each security outstanding in proportion to its share of the total value of all securities. However, no exactly equivalent index of market performance actually exists for the time period under consideration, although the new New York Stock Exchange Index provides a very good index of the returns on the market portfolio in recent times. The Standard & Poor 500 Composite Index, a value-weighted index, represents the closest approximation to such a measure that is available for the period covered by this study. Since these 500 securities represent the largest companies listed on the New York Stock Exchange, we use it as the best approximation available for the returns on our market portfolio M. Prior to March 1, 1957, the Standard & Poor Index was based on only ninety securities (fifty industrials, twenty rails, and twenty utilities), and hence for the earlier period the index is a poorer estimate of the returns on the market portfolio.

[67] Note that while most funds pay dividends on a quarterly basis, we treat all dividends as though they were paid as of December 31 only. This assumption, of course, will cause the measured returns on the fund portfolios on the average to be below what they would be if dividends were considered to be reinvested when received, but the data needed to accomplish this are not easily available. However, the

That is, the estimate of systematic risk for the jth fund is obtained from all the data available, and the number of sample observations varies from ten to twenty. Also, as was shown in Section III-B, under the assumption of Gaussian distributions β_{2j} is given by (3.12) and in the case of non-Gaussian Stable distributions β_{3j} is given by (3.27). The arguments in that section also indicate that to a very close approximation β_{2j} and β_{3j} are equal to the slope coefficient b_j in the market model.[68] Henceforth, we shall use the general notation β_j (without the following subscript) to denote the estimate of systematic risk, and the reader may interpret the measure either as β_{2j} or β_{3j}, depending on his inclination to accept the evidence regarding the infinite variance properties of the security returns (cf. references 4, 12, 19, 38, and 46).

The estimates of systematic risk, $\hat{\beta}_j$, for all 115 funds are given in Table 2 (col. 5) along with various other statistics which we shall discuss below. Figure 6 presents a frequency distribution of the coefficients. In addition, Table 3 presents a summary of the regression statistics for the sample of 115 funds.[69] We make no inferences from these statistics here except to note the following characteristics:

1) The average $\hat{\beta}$ coefficient for the 115 funds is only .840. This implies that the funds are fairly conservative in their investment policies—in general, offering investors portfolios with smaller systematic risks than the market portfolio (which implicitly represents a systematic risk of unity[70]). Hence, any attempt to

resulting bias should be quite small. In addition, the same bias is incorporated into the measured returns on the market portfolio.

[68] Since it was argued that the second term on the RHS of both (3.12) and (3.27) is trivially small, the question may arise as to whether this term will also be trivially small for a *portfolio* of securities. It will be, since the fraction x_j in (3.12) and (3.27) for a portfolio of K securities will on the average be equal to $1/N$, just as in the case of an individual security. To see this, let y_i be the fraction of the jth portfolio invested in the ith security and let h_i be the fraction of the ith security in the market portfolio. Then the portfolio disturbance term e_j is given by

$$e_j = \sum_{i=1}^{K} y_i h_i e_i' \,, \qquad (6.2)$$

where e_i' is the disturbance for the ith security. Under these conditions, the last term on the RHS of (3.12) is given by

$$\begin{aligned} \frac{x_j \sigma^2(e_j)}{\sigma^2(R_M)} &= \sum_{i=1}^{K} y_i h_i \sigma^2(e_i') \Big/ \sigma^2(R_M) \\ &\cong \sum_{i=1}^{K} y_i (1/N) \overline{\sigma^2(e')} \Big/ \sigma^2(R_M) \,, \end{aligned} \qquad (6.3)$$

where $\overline{\sigma^2(e_i')}$ is the average variance of the disturbance terms for the individual securities and $1/N$ is the average weight h_i. But

$$\sum_{i=1}^{K} y_i = 1 \,,$$

so

$$\frac{x_j \sigma^2(e_j)}{\sigma^2(R_M)} \cong \frac{1}{N} \overline{\sigma^2(e')} \Big/ \sigma^2(R_M) \,, \qquad (6.4)$$

where, as before, N is at least on the order of 1,000 and $\overline{\sigma^2(e')}$ is approximately the same size as $\sigma^2(R_M)$. (See the arguments given on p. 180.) The arguments for the non-Gaussian Stable case are analogous to these and need not be repeated.

[69] In the context of the market model discussed in Section III, the $\hat{a}$ in Table 3 is the intercept calculated as $\hat{a}_j = \overline{R}_j^* - \hat{\beta}_j \overline{R}_M^*$ and is presented only as a matter of information for the interested reader.

[70] We should mention that the errors in variables attenuation bias will cause our estimated coefficients to be smaller than the true coefficients, but it is very doubtful that these measurement errors are large enough to explain the total difference between .840 and 1.000. That is, let I_{Mt}, u_t, and b_j' be, respectively, the true index return, unbiased measurement error, and the true coefficient. Then, if we observe only $I_{Mt}^* = I_{Mt}' + u_t$, it is easily shown (cf. Johnston [33, chap. vi]) that

$$\begin{aligned} \text{Plim } \beta_j \cong \text{Plim } b_j &= \text{Plim } \frac{\hat{\text{cov}}\,(R_j^*, I_M^*)}{\sigma^2(I_M^*)} \\ &= \frac{b_j'}{1 + [\sigma^2(u)/\sigma^2(I_M')]} \,. \end{aligned}$$

Thus, the ratio of the variance of the measurement error to the variance of the true index, $[\sigma^2(u)/$

TABLE 2

MEASURES OF PERFORMANCE, δ^*, AND EFFICIENCY, γ^* (SEE P. 240 FOR DEFINITION OF γ^*), FOR 115 MUTUAL FUNDS IN THE PERIOD 1955–64 ALONG WITH VARIOUS OTHER STATISTICS (FUNDS ARE RANKED FROM HIGH TO LOW ON THE BASIS OF δ^*)

RANK (1)	ID. (2)	CODE (3)	$10^{R^*_{1964}}$ (4)	$\hat{\beta}$ (5)	$\hat{\beta}/\hat{r}$ (6)	$\hat{r}$ (7)	δ^* (8)	γ^* (9)
1	1186	0	1.402	0.538	0.762	0.706	0.627	0.427
2	176	0	1.469	0.954	1.074	0.888	0.323	0.216
3	2186	0	1.231	0.792	0.918	0.863	0.229	0.117
4	169	4	1.138	0.712	0.750	0.949	0.207	0.173
5	225	2	0.937	0.490	0.554	0.885	0.204	0.147
6	267	4	1.083	0.656	0.740	0.886	0.202	0.127
7	162	0	1.225	0.819	1.005	0.815	0.199	0.034
8	250	0	1.168	0.762	0.809	0.942	0.194	0.152
9	2262	2	0.955	0.548	0.583	0.940	0.171	0.139
10	246	2	0.961	0.566	0.603	0.938	0.161	0.127
11	234	0	1.235	0.878	0.940	0.934	0.156	0.101
12	206	2	1.088	0.716	0.754	0.950	0.154	0.121
13	192	3	1.144	0.790	0.821	0.962	0.144	0.116
14	227	4	1.019	0.660	0.697	0.947	0.135	0.102
15	233	3	1.102	0.758	0.856	0.886	0.131	0.044
16	151	2	0.946	0.593	0.668	0.888	0.122	0.055
17	2268	2	0.891	0.537	0.555	0.968	0.116	0.100
18	207	3	0.989	0.671	0.754	0.890	0.095	0.022
19	260	2	0.773	0.435	0.528	0.824	0.089	0.007
20	175	4	1.067	0.768	0.793	0.969	0.087	0.065
21	221	0	1.240	0.970	1.094	0.887	0.080	-0.030
22	142	2	0.822	0.508	0.634	0.801	0.074	-0.039
23	215	2	0.848	0.548	0.645	0.850	0.064	-0.023
24	141	0	1.154	0.892	1.052	0.848	0.063	-0.079
25	218	2	0.809	0.512	0.554	0.924	0.057	0.020
26	152	4	1.086	0.828	0.865	0.957	0.052	0.019
27	144	3	1.116	0.865	0.880	0.983	0.050	0.036
28	201	2	0.854	0.586	0.623	0.941	0.036	0.003
29	177	2	0.831	0.562	0.596	0.943	0.034	0.004
30	257	2	0.850	0.585	0.614	0.952	0.033	0.007
31	168	2	0.857	0.594	0.627	0.948	0.031	0.002
32	1268	2	0.954	0.706	0.732	0.964	0.029	0.005
33	157	0	1.013	0.774	1.035	0.748	0.028	-0.205
34	256	4	1.194	0.987	1.006	0.981	0.019	0.002

TABLE 2—*Continued*

35	226	2	0.779	0.523	0.600	0.872	0.017	-0.052
36	185	3	1.198	0.995	1.015	0.980	0.016	-0.002
37	219	4	1.032	0.815	0.849	0.960	0.009	-0.021
38	203	1	0.490	0.208	0.335	0.620	0.009	-0.105
39	235	1	1.083	0.875	0.929	0.942	0.007	-0.041
40	163	4	1.091	0.890	0.996	0.894	0.002	-0.092
41	200	3	1.146	0.952	0.990	0.962	0.002	-0.031
42	232	4	1.046	0.840	0.857	0.980	0.002	-0.013
43	208	3	1.051	0.846	0.870	0.972	0.001	-0.021
44	236	2	0.923	0.704	0.749	0.940	-0.000	-0.040
45	1000	0	1.081	0.890	0.917	0.971	-0.008	-0.032
46	259	3	0.994	0.796	0.870	0.915	-0.011	-0.077
47	2191	2	0.888	0.685	0.741	0.924	-0.018	-0.068
48	140	0	1.080	0.907	0.954	0.951	-0.024	-0.066
49	178	3	1.017	0.838	0.880	0.952	-0.026	-0.063
50	245	3	0.999	0.827	0.861	0.960	-0.034	-0.064
51	240	2	0.799	0.603	0.639	0.944	-0.034	-0.066
52	1212	1	0.511	0.286	0.407	0.703	-0.040	-0.148
53	1211	1	0.721	0.527	0.617	0.854	-0.045	-0.125
54	241	3	1.093	0.946	0.981	0.964	-0.046	-0.078
55	180	3	1.056	0.912	0.939	0.971	-0.052	-0.077
56	216	0	1.181	1.058	1.115	0.949	-0.058	-0.108
57	2212	1	0.611	0.419	0.541	0.774	-0.058	-0.167
58	2261	2	0.800	0.635	0.671	0.947	-0.062	-0.093
59	217	3	1.088	0.962	0.974	0.988	-0.065	-0.075
60	252	4	1.077	0.949	0.982	0.966	-0.065	-0.095
61	165	2	0.780	0.626	0.683	0.916	-0.073	-0.125
62	188	2	0.808	0.662	0.707	0.936	-0.078	-0.118
63	251	3	1.112	1.004	1.046	0.960	-0.079	-0.116
64	239	2	1.173	1.089	1.245	0.875	-0.093	-0.232
65	153	3	1.059	0.962	0.991	0.971	-0.094	-0.120
66	190	4	1.059	0.962	1.015	0.948	-0.094	-0.141
67	172	0	1.010	0.907	0.976	0.929	-0.094	-0.156
68	2159	3	0.919	0.807	0.831	0.971	-0.096	-0.117
69	1209	4	1.034	0.944	0.977	0.966	-0.103	-0.132
70	150	3	0.948	0.855	0.869	0.984	-0.109	-0.122
71	166	2	0.793	0.687	0.713	0.964	-0.116	-0.138
72	244	2	0.816	0.728	0.760	0.958	-0.128	-0.157
73	1159	0	0.942	0.878	0.918	0.956	-0.136	-0.172
74	202	3	0.979	0.920	0.966	0.952	-0.136	-0.178

TABLE 2—*Continued*

75	189	0	1.086	1.049	1.123	0.934	-0.145	-0.211
76	197	1	0.894	0.845	0.906	0.933	-0.155	-0.209
77	145	4	0.881	0.834	0.868	0.961	-0.158	-0.188
78	195	2	0.721	0.664	0.695	0.955	-0.167	-0.195
79	231	3	1.050	1.036	1.069	0.969	-0.169	-0.199
80	187	3	1.061	1.051	1.072	0.980	-0.171	-0.191
81	243	3	0.961	0.940	0.970	0.969	-0.172	-0.199
82	193	3	0.855	0.822	0.867	0.948	-0.173	-0.214
83	249	0	1.062	1.060	1.137	0.932	-0.179	-0.248
84	1148	2	0.809	0.777	0.841	0.924	-0.180	-0.237
85	147	2	0.852	0.834	0.856	0.974	-0.187	-0.207
86	2148	0	0.934	0.934	1.065	0.877	-0.194	-0.310
87	263	4	0.802	0.806	0.898	0.898	-0.212	-0.294
88	174	2	0.809	0.821	0.858	0.957	-0.218	-0.251
89	182	3	0.998	1.034	1.064	0.972	-0.219	-0.246
90	173	0	1.055	1.108	1.252	0.885	-0.228	-0.356
91	194	0	0.874	0.919	1.038	0.885	-0.241	-0.347
92	171	3	0.951	1.011	1.055	0.958	-0.245	-0.285
93	210	0	1.289	1.396	1.490	0.937	-0.251	-0.334
94	253	0	0.993	1.065	1.126	0.946	-0.252	-0.306
95	160	3	0.896	0.965	1.058	0.912	-0.260	-0.342
96	167	3	0.857	0.933	0.965	0.967	-0.270	-0.298
97	1261	3	0.879	0.965	1.039	0.929	-0.277	-0.343
98	1223	0	0.983	1.091	1.195	0.913	-0.285	-0.378
99	247	0	0.787	0.884	0.957	0.924	-0.296	-0.361
100	2223	1	0.792	0.890	0.928	0.959	-0.297	-0.330
101	255	0	0.815	0.923	1.056	0.874	-0.304	-0.422
102	254	1	0.789	0.934	0.977	0.956	-0.339	-0.377
103	164	0	0.748	0.892	0.986	0.905	-0.342	-0.426
104	155	0	0.736	0.890	0.950	0.937	-0.353	-0.406
105	220	2	0.666	0.821	0.867	0.947	-0.361	-0.402
106	2211	0	1.023	1.229	1.439	0.854	-0.368	-0.555
107	224	1	0.893	1.085	1.164	0.932	-0.370	-0.440
108	146	0	0.699	0.876	0.947	0.925	-0.378	-0.441
109	184	3	0.900	1.118	1.138	0.982	-0.392	-0.410
110	2209	0	1.043	1.278	1.330	0.961	-0.392	-0.438
111	222	4	0.945	1.185	1.214	0.976	-0.407	-0.433
112	158	0	0.678	1.048	1.104	0.949	-0.552	-0.602
113	205	3	0.976	1.402	1.755	0.799	-0.569	-0.883
114	198	3	0.708	1.262	1.297	0.973	-0.712	-0.743
115	1191	0	0.333	1.409	1.665	0.846	-1.219	-1.447

NOTE.—For Wiesenberger classification code, see n. 1 to Table 1 and n. 79 in text.

TABLE 3

SUMMARY OF REGRESSION STATISTICS FOR THE SAMPLE OF 115 MUTUAL FUNDS

ITEM	MEAN VALUE	MEDIAN VALUE	EXTREME VALUES		MEAN ABSOLUTE DEVIATION[a]
			Minimum	Maximum	
$\hat{\alpha}$	−.007	−.006	−.088	.030	.020
$\hat{\beta}$	.840	.846	.208	1.409	.167
$\hat{r}$	.923	.943	.620	.988	.046
$\rho(e^*_t, e^*_{t-1})$[b]	−.063	−.033	−.699	.590	.213
n	17.0	19.0	10.0	20.0	3.12

[a] Defined as

$$\sum_{i=1}^{115} |\bar{x} - x_i| \Big/ 115 .$$

[b] First-order autocorrelation of residuals. The average $\hat{\rho}^2$ is .074.

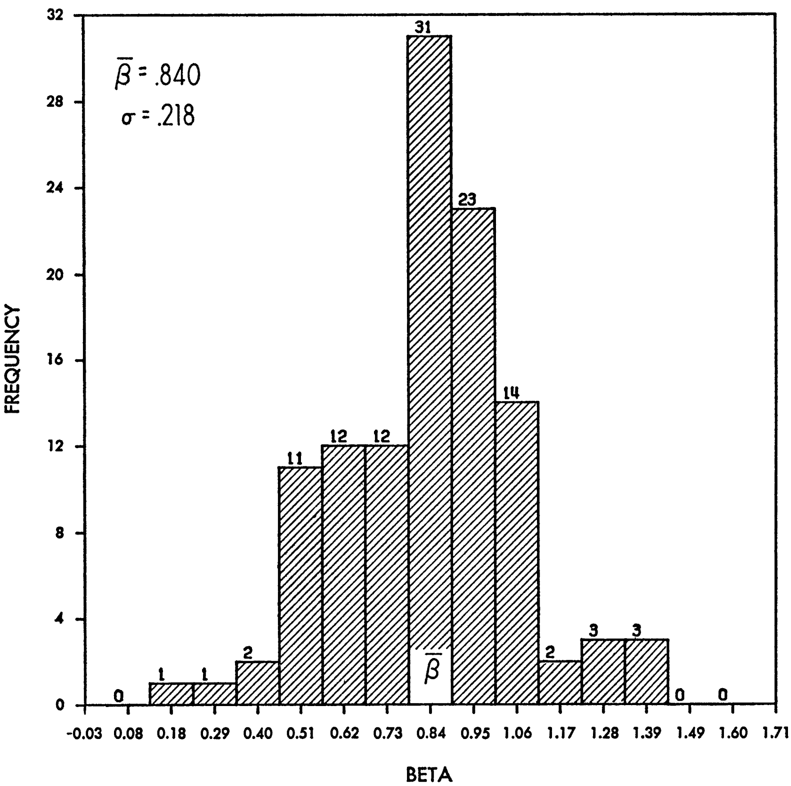

FIG. 6.—Frequency distribution (half-sigma intervals) of the estimates of systematic risk β for 115 mutual funds using all data available in the period 1945–64.

compare the performance of mutual funds to such a market index without explicitly allowing for the trade-off between higher risks and higher returns will be biased against the funds.

2) The correlation coefficients listed in column 7 of Table 2 (and for which a frequency distribution is presented in Figure 7) are in general quite high (with an average of .923), indicating that the funds on the average hold well-diversified portfolios. The median value (see Table 3) is 0.943. In conjunction with this, we note that the average mean absolute deviation of the residuals is .038 and the average standard deviation of the residuals, $\hat{\sigma}(e)$, is .052.

3) It is of special interest to note in Table 3 that the average first-order autocorrelation of the residuals, $\rho(e_t^*, e_{t-1}^*)$, is only $-.063$. Thus, it would seem that on the average the model is well specified with regard to this factor. However, it should be noted that there are some relatively large extreme values, namely, a minimum of $-.699$ and a maximum of $+.590$. But with such small sample sizes, the standard error of estimate of the serial correlation coefficient is quite large,

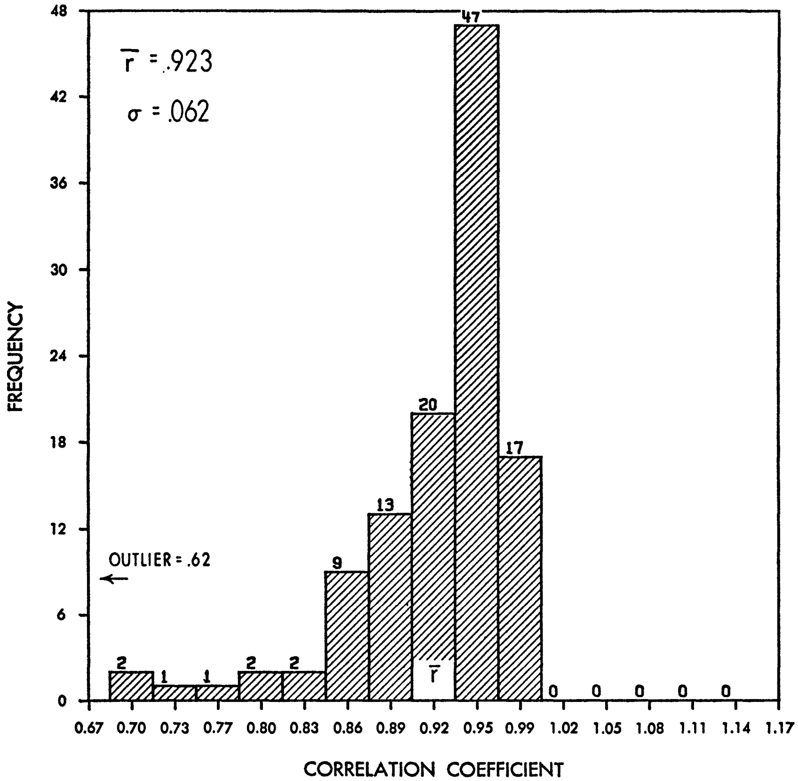

FIG. 7.—Frequency distribution (half-sigma intervals) of the correlation coefficients r_j between the returns on 115 mutual funds and the returns on the market portfolio M in the period 1945–64.

$\sigma^2(I'_M)$], would have to be approximately 19 per cent in order to explain the average $\hat{\beta}$ of .840 if the true average $\beta \cong b'$ was 1.

Note that any measurement errors in the index R_M^* will cause consistent underestimation of the systematic risk of the portfolios and will therefore tend to cause the portfolio's performance to appear better than it actually is.

and these observations are within the range of what we could expect given a sample of 115 funds.[71]

Since the empirical evaluation tests to follow will be crucially dependent on the estimates of β_j obtained from these "regressions," it is extremely important that the model be well specified and stationary through time and that the estimates of the parameter β be invariant to the length of the time interval over which the sample returns are calculated. The remainder of this section is devoted to an evaluation of the estimates of the market model with specific reference to these problems.

In order to test the model, the following scatter diagrams were calculated for every tenth fund in the sample: (1) fund return versus market return, (2) residual versus market return, (3) residual in $t+1$ versus residual in t, and (4) residual versus time. The diagrams for the Colonial Fund, which is fairly typical of the sample as a whole, are given in Figure 8. In general, for the sample as a whole, panel *a* indicates that the linearity assumption is valid, and panel *b* indicates that the residuals appear to be uncorrelated with the market returns. There is some slight evidence,[72] as in panel *d* of Figure 8, that the model may not be stationary through time for all funds. We shall present more evidence on this point below after consideration of the invariance of the estimates to the length of the time interval over which the returns are calculated.

Stability of the estimate of systematic risk.—It was pointed out in Section IV-C that the solution to the horizon problem implies that the estimate of the risk coefficient, β, will be invariant to the length of the time interval over which returns are measured—as long as the market horizon H is close to zero and the returns are stated as continuously compounded rates. That is, let ${}_N\hat{\beta}_j$ be the risk coefficient for the jth fund estimated from a time series of N-period returns[73] (properly transformed by [4.10]); then

$$\beta_j = \sum_{t=1}^{n_j}({}_NR^*_{jt} - {}_N\bar{R}^*_j)({}_NR^*_{Mt} - {}_N\bar{R}^*_M) \Big/ \sum_{t=1}^{n_j}({}_NR^*_M - {}_N\bar{R}^*_M)^2$$

where the prescript N indicates the length of the time interval over which each of the sample returns is calculated. The arguments in Section IV imply that the market horizon period is very likely to be nearly instantaneous, and under these conditions the estimates based on the natural logarithms of the observed return data will have the following property: ${}_1\hat{\beta}_j = {}_2\hat{\beta}_j = \ldots = {}_N\hat{\beta}_j$.

In order to test the validity of these arguments, the risk coefficients ${}_2\beta_j$ were estimated from ten observations of two-year returns, ${}_2R^*_j$, for those fifty-six funds in the sample having a full twenty years of data available.[74] These estimates are given in column 3 of Table 4 along with the estimates based on annual data, ${}_1\hat{\beta}_j$, in column 2. The differences between the coefficients are given in column 4 of Table 4. Unfortunately, it is difficult to specify formal tests of the differences, since the errors are certainly not independent. Hence, we are forced to rely on

[71] The t statistic (cf. Johnston [33, p. 33]) for testing the significance of ρ is given by

$$t(n-2) = \frac{\rho[n-2]^{1/2}}{[1-\rho^2]^{1/2}}.$$

[72] Which, given the small sample sizes, is very weak.

[73] As opposed to H-period returns.

[74] Of course, the returns on the market portfolio, ${}_2R^*_M$, were also translated to the two-year dimension.

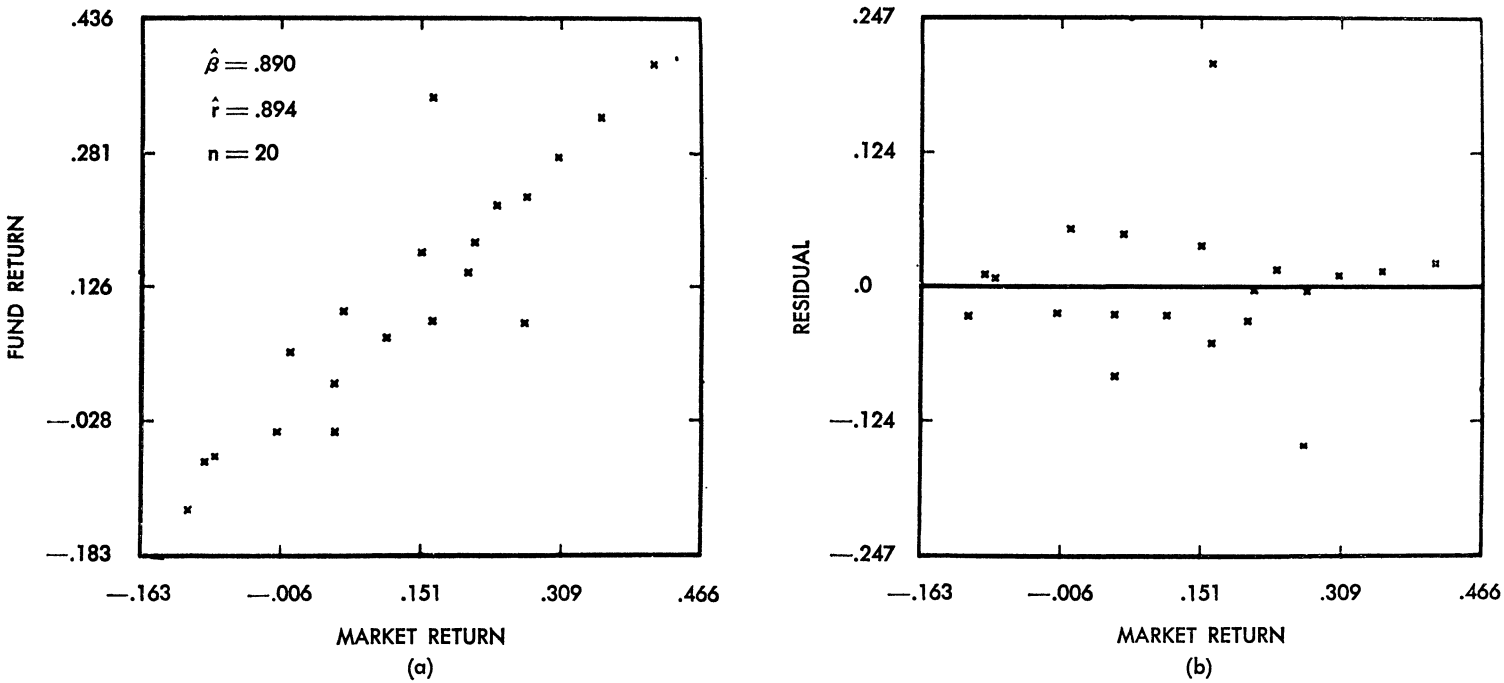

FIG. 8.—Scatter diagrams for testing the adequacy of the assumptions of the market model for the Colonial Funds Inc. (identification number 163)

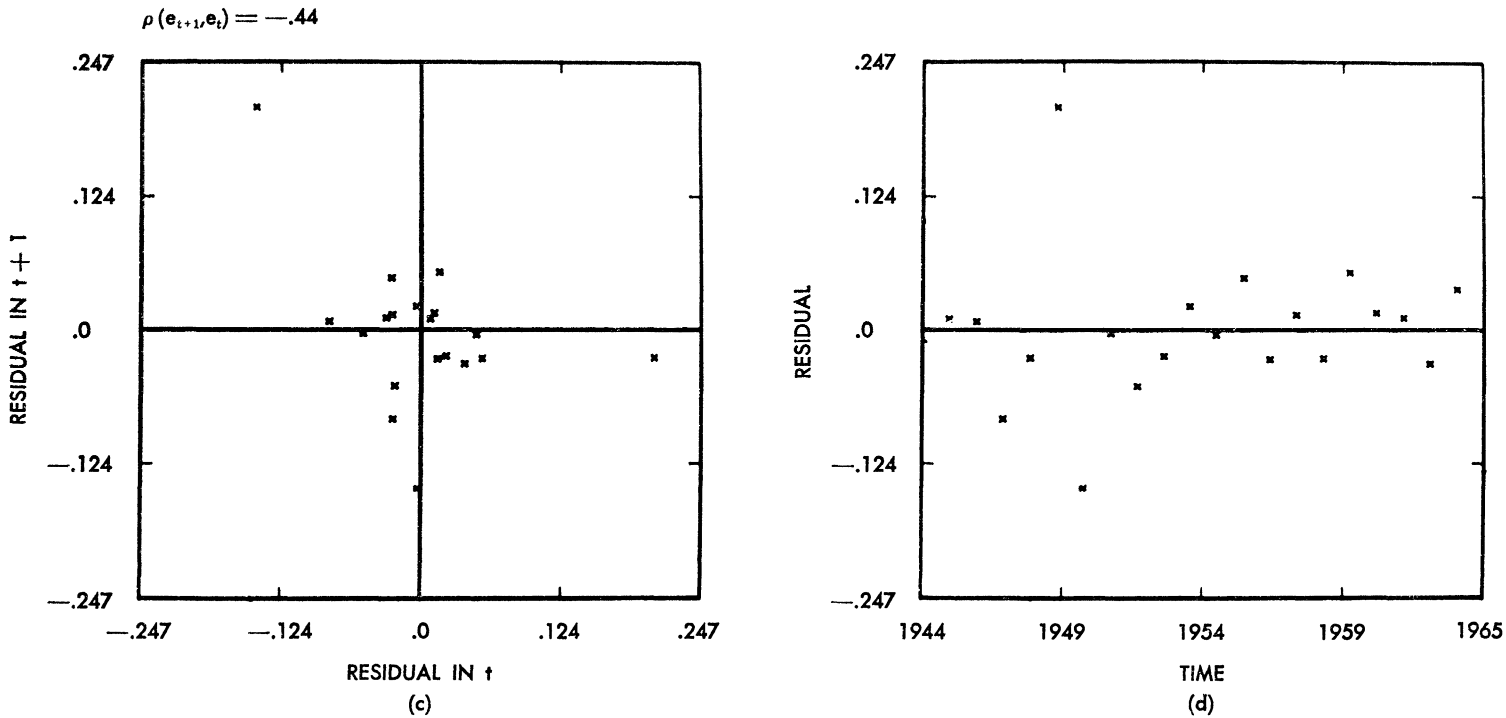

FIG. 8.—*Continued*

TABLE 4

COMPARISON OF ESTIMATES OF SYSTEMATIC RISK OBTAINED FROM ONE- AND TWO-PERIOD DATA FOR FIFTY-SIX FUNDS

ID NUMBER	$_1\hat{\beta}$	$_2\hat{\beta}$	$(_1\hat{\beta} - {}_2\hat{\beta})$
140	0.907	0.976	-0.069
141	0.892	0.696	0.196
142	0.508	0.246	0.263
145	0.834	0.681	0.154
147	0.834	0.769	0.064
1148	0.777	0.959	-0.182
2148	0.934	1.203	-0.269
151	0.593	0.544	0.049
152	0.828	0.988	-0.160
153	0.962	1.147	-0.185
157	0.774	0.396	0.378
162	0.819	1.090	-0.271
163	0.890	1.011	-0.121
166	0.687	0.699	-0.012
169	0.712	0.633	0.079
171	1.011	0.930	0.081
174	0.821	0.870	-0.049
175	0.768	0.784	-0.016
177	0.562	0.689	-0.127
178	0.838	1.020	-0.182
180	0.912	0.811	0.101
182	1.034	1.082	-0.048
184	1.118	1.145	-0.027
185	0.995	1.044	-0.048
187	1.051	1.040	0.011
188	0.662	0.792	-0.130
190	0.962	0.958	0.004
2191	0.685	0.559	0.126
195	0.664	0.698	-0.034
198	1.262	1.486	-0.224
200	0.952	0.866	0.086
201	0.586	0.612	-0.026
205	1.402	1.209	0.193
215	0.548	0.520	0.028
216	1.058	0.962	0.096
217	0.962	1.107	-0.145
219	0.815	0.766	0.049
220	0.821	0.762	0.060
221	0.970	1.080	-0.110
222	1.185	1.097	0.087
2223	0.890	0.775	0.115
225	0.490	0.457	0.033
226	0.523	0.609	-0.086
233	0.758	0.661	0.097
236	0.704	0.645	0.059
240	0.603	0.580	0.024
241	0.946	0.990	-0.044
243	0.940	0.987	-0.047
245	0.827	0.784	0.043
251	1.004	1.073	-0.070
252	0.949	0.961	-0.012
257	0.585	0.578	0.007
259	0.796	0.671	0.125
260	0.435	0.311	0.124
2261	0.635	0.590	0.045
263	0.806	0.874	-0.068

$_1\bar{\hat{\beta}} = .8301$

$_2\bar{\hat{\beta}} = .8299$

Mean absolute difference $= \dfrac{\sum_{i=1}^{56} \left| {}_1\hat{\beta}_i - {}_2\hat{\beta}_i \right|}{56} = .10$

informal descriptive measures of the relationships. Figure 9 presents a plot of $_2\hat{\beta}_j$ against $_1\hat{\beta}_j$, and the reader will note that a 45-degree line would represent perfect correspondence between the two estimates. The correlation between the two estimates is .89. For the fifty-six funds the averages for the two estimates are $_1\bar{\beta} = .8301$ and $_2\bar{\beta} = .8299$, and thus the averages differ by only .0002. The mean absolute difference between the two estimates for the fifty-six funds is only .10. Given the differences in sample sizes, these differences seem small, and they seem to support the implications of the theory and the assumption of an instantaneous horizon period quite well. We shall see below that the linearity of the opportunity set implied by equation (4.11) also seems to indicate that the log transformation is appropriate. But for the moment, we accept the log form as appropriate and proceed to an examination of the stationarity of the risk measure through time. Also, since the evidence indicates that the estimates are stable and $_N\hat{\beta}_j = \hat{\beta}_j$, we henceforth drop the preceding subscript N and refer to the measure of risk only as β_j.

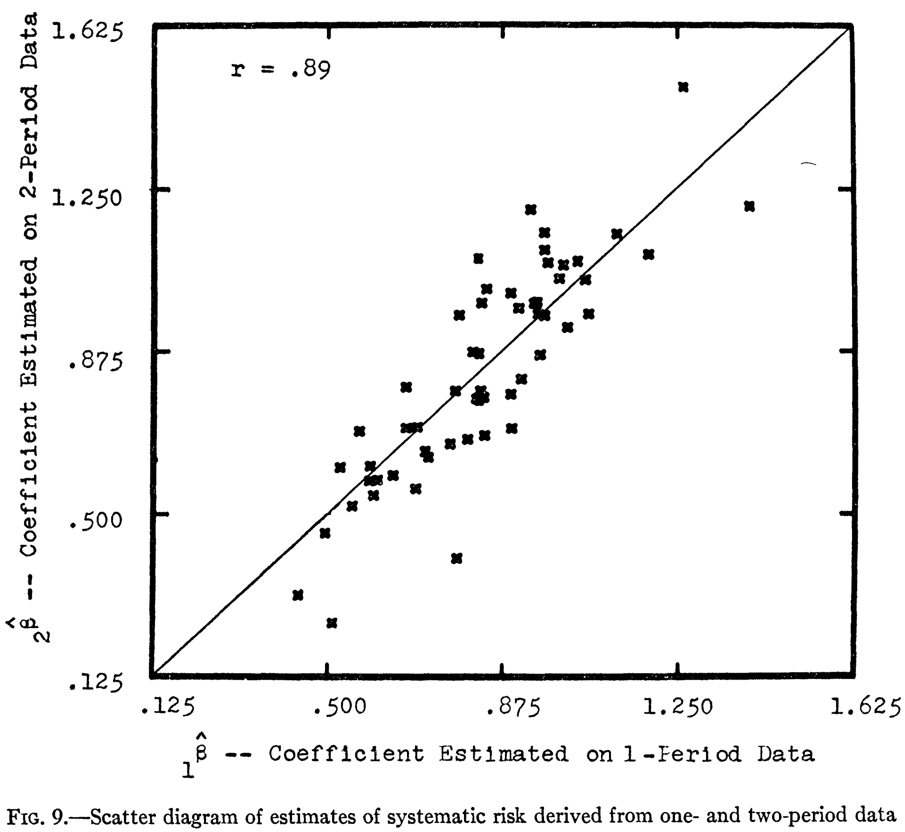

FIG. 9.—Scatter diagram of estimates of systematic risk derived from one- and two-period data

Stationarity of the measure of systematic risk.—If the concept of systematic risk is to be of practical use in evaluating and selecting portfolios, it must be stationary through time. That is, the investor selecting a portfolio must be able to use past historical data to obtain estimates which will be a good indication of future risk. Furthermore, in evaluating portfolios, we must be able to assume that the riskiness of the portfolio has not changed over the period under consideration. Blume [4], in a detailed examination of the market model, finds that for 251 individual securities, the coefficients b_j (approximately equal to our β_j) are approximately stationary over the thirty-four-year period 1927–60. This is an extremely important result and indicates that the systematic risk of a portfolio of securities (each representing a constant fraction of the portfolio) will also be stationary through time. However, this result is not sufficient to establish the stationarity of risk in a sample of fully managed portfolios such as the mutual funds under examination here. If the managers so choose, they can substantially change the level of systematic risk in either direction by altering the proportions of high- and low-risk securities in the portfolio.

In order to test the stationarity of the riskiness of the mutual funds, the twenty-year period 1945–64 was split into two parts. The risk coefficient β_j was then estimated in each of the two ten-year periods for the fifty-six funds having a complete twenty years of data available. The estimates $\hat{\beta}_{j,45-54}$ and $\hat{\beta}_{j,\,55-64}$ for all fifty-six funds are given in Table 5 along with the difference between the estimates. The average coefficient in the latter period was .803, as compared with .864 for the earlier period. The mean absolute difference between the estimates is .11 and seems quite small. A summary of the average values of $\hat{\beta}$, $\hat{r}$, $\hat{\rho}$, and $\hat{\rho}^2$ is given in Table 6, and the reader will note that, except for the autocorrelation of residuals, the average coefficients are quite similar in the two sample periods.

Figure 10 presents a scatter diagram of $\hat{\beta}_{55-64}$ against $\hat{\beta}_{45-54}$, and again it should be noted that a 45-degree line represents perfect correspondence between the two estimates. The correlation between the two estimates is[75] .74 and while the relationship is not perfect, the scatter does show evidence that the funds tend to maintain their level of riskiness through time. A detailed examination of Figure 10 indicates that (1) the marginal variance of the estimates for the earlier period seems larger and (2) there seems to be a hint of curvilinearity. In addition, an examination of Table 5 indicates thirty-four observations for which the difference $(\hat{\beta}_{55-64} - \hat{\beta}_{45-54})$ is negative. Under the assumption that the two coefficients are actually identical, the expected number of negative values in column 4 of Table 5 is 28. Using the normal approximation to the binomial distribution, the standard deviation of the distribution is $\sigma = \sqrt{npq} = \sqrt{(56)(.5)(.5)} = 3.75$, and the excess number of negative values is thus within 2 S.D. of the expected number. Given these results, we shall proceed under the assumption that the risk coefficients are stationary. Hence, in the following evaluation of fund portfolios, the risk coefficient is estimated from all data available on each fund in order to minimize the sampling error in the estimates.

Summarizing the empirical results so far we have found that the evidence supports: (1) the assumptions of the market model regarding (*a*) the linear relationship between fund returns and the market factor and (*b*) the independence of

[75] The correlation between the estimates is reduced considerably by the presence of the outlier in the middle right hand side of the figure. This point represents the Investment Trust of Boston and is caused entirely by one observation. Wiesenberger's data indicates the fund earned 177 per cent in 1945 and while suspect, the author was unable to confirm this to be an error. Thus the observation was left in the sample.

TABLE 5

COMPARISON OF ESTIMATES OF SYSTEMATIC RISK IN THE TWO TEN-YEAR PERIODS 1945–54 AND 1955–64 FOR FIFTY-SIX FUNDS

ID NUMBER	$\hat{\beta}_{55-64}$	$\hat{\beta}_{45-54}$	$(\hat{\beta}_{55-64} - \hat{\beta}_{45-54})$
140	0.935	0.902	0.033
141	0.729	1.071	-0.342
142	0.394	0.637	-0.243
145	0.869	0.812	0.057
147	0.789	0.885	-0.097
1148	0.651	0.902	-0.252
2148	0.722	1.192	-0.469
151	0.512	0.683	-0.171
152	0.759	0.897	-0.138
153	0.888	1.040	-0.152
157	0.819	0.727	0.091
162	0.890	0.743	0.146
163	0.881	0.914	-0.033
166	0.648	0.723	-0.076
169	0.751	0.677	0.074
171	1.037	1.001	0.036
174	0.726	0.915	-0.189
175	0.776	0.769	0.006
177	0.548	0.577	-0.029
178	0.865	0.798	0.067
180	0.909	0.927	-0.018
182	1.049	1.004	0.045
184	1.104	1.136	-0.033
185	1.050	0.953	0.097
187	1.024	1.082	-0.058
188	0.652	0.677	-0.025
190	0.900	1.047	-0.146
2191	0.689	0.711	-0.022
195	0.657	0.664	-0.006
198	1.179	1.328	-0.149
200	0.985	0.934	0.051
201	0.606	0.572	0.035
205	0.998	1.762	-0.764
215	0.524	0.575	-0.051
216	1.027	1.099	-0.072
217	0.970	0.953	0.017
219	0.842	0.811	0.031
220	0.805	0.835	-0.030
221	0.864	1.090	-0.225
222	1.109	1.280	-0.171
2223	0.826	0.970	-0.144
225	0.482	0.515	-0.033
226	0.377	0.661	-0.284
233	0.816	0.683	0.133
236	0.735	0.684	0.051
240	0.665	0,550	0.115
241	0.997	0.907	0.090
243	0.944	0.940	0.003
245	0.879	0.768	0.111
251	0.961	1.055	-0.094
252	0.921	0.983	-0.062
257	0.584	0.595	-0.012
259	0.813	0.793	0.020
260	0.422	0.472	-0.050
2261	0.664	0.610	0.054
263	0.746	0.888	-0.143

$\bar{\beta}_{45-54}$ = .864

$\bar{\beta}_{55-64}$ = .803

Mean absolute difference $= \dfrac{\sum_{i=1}^{56} |\hat{\beta}_{i,45-54} - \hat{\beta}_{i,55-64}|}{56} = .11$

residuals and the market factor; (2) the theoretical argument regarding the stability of the estimate of systematic risk; and (3) the assumption of stationary risk levels for fund portfolios. The first result indicates that our estimates of systematic risk are valid. The second result regarding the invariance of the estimates to the length of the time interval over which the sample returns are calculated indicates that the risk coefficients may be used for a horizon interval of any length. Finally, the third result indicates that the future risk of a portfolio may be estimated from past data and that in general a more efficient estimate of a portfolio's risk may be obtained by using all past data.[76] Thus, we now continue

TABLE 6

AVERAGE VALUES OF SELECTED STATISTICS ASSOCIATED WITH THE ESTIMATES OF SYSTEMATIC RISK IN THE TWO TEN-YEAR PERIODS

ITEM	SAMPLE PERIOD	
	1955–64	1945–54
$\hat{\beta}$	.803	.864
$\hat{r}^2$	.894	.863
$\hat{\rho}(e_t,e_{t-1})$	−.164	−.068
$\hat{\rho}^2(e_t,e_{t-1})$	.120	.051

[76] Of course, in applying the model to any particular portfolio, the investor who believes that the risk of that portfolio has changed will be well advised to devote the necessary resources to obtaining recent data (on a monthly, weekly, or daily basis) in order to obtain estimates of the present riskiness of the portfolio.

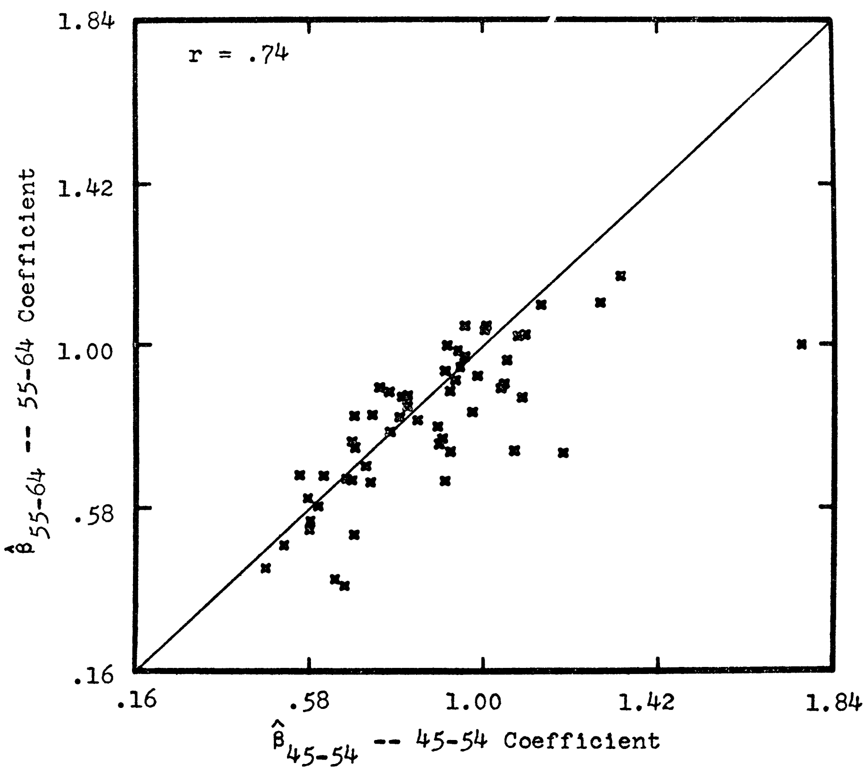

FIG. 10.—Scatter diagram of the estimated risk coefficients obtained for fifty-six funds in the two ten-year periods 1945–54 and 1955–64.

to a consideration of the performance of mutual fund portfolios.

B. THE EVALUATION OF MUTUAL FUND PORTFOLIOS

The hypotheses.—We turn now to a brief review of the questions originally posed in the Introduction before we moved to a detailed analysis of the underlying issues. The reader will recall that these questions involved (1) an examination of the hypothesis of the predominance of risk aversion among investors and the validity of the concept of systematic risk as implied by the capital assets pricing model and (2) an evaluation of the historical performance of mutual fund portfolios with specific regard to the ability of mutual fund managers to predict future security prices.

1. *Risk aversion and the measure of systematic risk.*—There are two essential issues involved here, and it is conceptually impossible to test both simultaneously. That is, we would like to know (1) whether the security markets are dominated by investors who are averse to risk and (2) whether the coefficient β is a valid measure of risk. Clearly now, we must assume that one of these is true in order to test the other. If the market is not dominated by risk averters, no measure of "risk" will be positively related to returns. On the other hand, if we do not have an "appropriate" measure of risk, the absence of a positive relationship between risk and return implies nothing about the predominance of risk aversion in the capital markets. However, since we continually observe people behaving as though they were averse to risk[77] (i.e., generally holding diversified multi-asset portfolios and buying insurance), we shall assume the former and test the latter.[78]

Unfortunately, it proves difficult to specify formal tests of the measure of risk β. Thus, we must be content at this point to judge its adequacy in terms of its apparent consistency with the implications of the assumption of dominant risk aversion on the part of investors. For instance, we would conclude that the measure was inconsistent with our assumption of risk aversion if we found that a plot of β versus realized returns on the fund portfolios yielded a negative or zero slope in a period during which re-

[77] At least risk aversion is generally observed when the risk of substantial losses exists (as there most certainly is in the case of non-trivial investments in securities). However, there appears to be some stituations, usually involving a high probability of small losses in conjunction with a small probability of large gains, in which people often behave as though they were risk lovers. For a discussion of these points, see Friedman and Savage [25] and Markowitz [41].

[78] We might note at this point that Latané [35] and Sharpe [54] have found evidence of a positive relationship between risk and return in the capital markets. Latané examined the differences in returns earned on common stocks, commodities, and bonds and finds a positive relationship between the riskiness and the returns on these instruments. Sharpe examined the relation between risk (measured by the standard deviation of annual returns) and the arithmetic mean of annual returns on mutual fund portfolios over the period 1954–63. We note here, however, that this type of test is subject to several deficiencies. As M. Miller has pointed out, any tests involving the relationship between the arithmetic mean and second moment of the distribution of returns are positively biased if the distribution of returns is positively skewed. Indeed, as shown by Cramer [10, p. 348], the sample covariance between the mean, $\bar{x}$, and second moment μ_2, of any distribution is

$$\text{cov}\,(\bar{x},\mu_2) = \frac{n-1}{n^2}\,\mu_3\,,$$

where μ_3 is the third moment of the distribution and n is the sample size. Since the distributions of annual returns must be skewed (since the maximum value is $+\infty$ and the minimum is zero), we know $\mu_3 > 0$, and any attempt to test the assumption of risk aversion by examining the relationship between the arithmetic average and σ is subject to this bias.

alized market returns were above the riskless rate.

Figure 11 presents a scatter diagram of the returns (measured net of management expenses but gross of loading charges) of the 115 funds plotted against their respective $\hat{\beta}$ coefficients calculated by (6.1). Thus, for the ten-year holding period 1955–64, Figure 11 represents the empirical counterpart[79] of the performance measure diagrammed in Figure 4. The returns plotted on the ordinate are the natural logarithm of the ten-year wealth relatives (assuming reinvestment of dividends at the end of the year). In terms of our previous notation,

[79] The symbols with which the fund portfolios are plotted denote the Wiesenberger classification

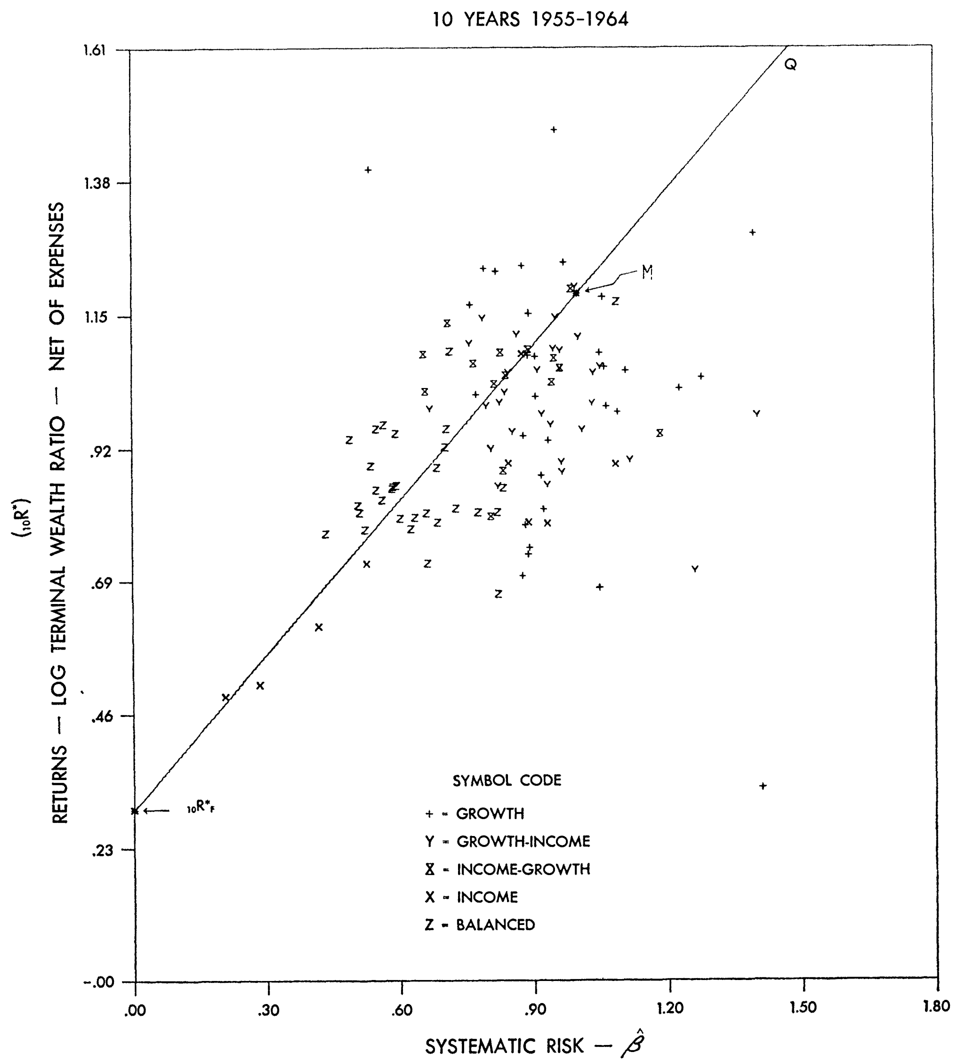

FIG. 11.—Scatter diagram of risk and (net) return for 115 open-end mutual funds in the ten-year period 1955–64.

$$_{10}R^*_{j,1964} = \log_e (1 + {_{10}R_{j,1964}}) = \log_e \frac{W_{j,1964}}{W_{j,1955}}$$

$$= \log_e \prod_{t=1955}^{1964} (1 + {_1R_{j\,t}}) \, .$$

The returns $_{10}R^*_{j,1964}$ for all of the funds are listed in column 4 of Table 2 along with the Wiesenberger classification code (col. 3). The points $_{10}R^*_F$ and M in Figure 11 are, respectively, the ex post experience of the risk-free asset and the market portfolio. The ten-year risk-free return is $_{10}R^*_F = \log_e(1 + {_{10}R_F}) = \log_e(1 + .03)^{10} = .296$, where 3.0 per cent was the yield to maturity of a ten-year government bond in 1955.[80] The estimated return on the market portfolio, $_{10}R^*_M$, was 1.187 in this period.[81] Thus, the market line $_{10}R^*_F MQ$ in Figure 11 given by (6.5),

$$E({_{10}R^*} \mid {_{10}R^*_M}, \hat{\beta}) = .296 + (1.187 - .296)\beta = .296 + .891\beta \, , \tag{6.5}$$

represents the possible expected combinations of systematic risk and return conditional on the *actual* realized returns on the market portfolio which were available to an investor with a ten-year planning horizon in 1955.

The patterns observed in Figure 11 seem to confirm the adequacy of our measure of risk. We observe a positive relationship between the realized returns on these portfolios and their systematic risk,[82] which is exactly what we would predict if: (1) investors were averse to risk, demanding premiums (in the form of higher expected returns) for accepting

of the investment objectives of each fund based upon its stated investment emphasis (see key to Fig. 11). They are plotted in this manner in order to illustrate the correspondence between the fund managers' statements regarding the objectives of the fund (the basis of the Wiesenberger classifications) and the measure of systematic risk.

Wiesenberger (1961 edition, p. 134) defines *growth* funds to be those whose primary objective is the long-term growth of capital and for which the "risk of price depreciation in declining periods is normally higher . . . than for many others." *Growth-income* funds are those which combine an emphasis on long-term growth of capital with a consideration of "income and/or relative stability." *Income-growth* funds combine an emphasis on current income with the possibility of long-term capital growth. *Income* funds are defined as those whose "primary objective is the most generous possible current income," and the *balanced* funds are those which "place more emphasis on relative stability and continuity of income than do those in the preceding groups." On the basis of these definitions, it is our guess that the classifications as listed in Fig. 11 represent declining riskiness from growth to balanced.

While the patterns of Fig. 11 seem to lend some credence to the belief that the fund managers do have an idea of the amount of risk their portfolios contain, the classifications for individual funds are certainly not perfectly consistent with the measure of systematic risk. Table 7 presents the average and median $\hat{\beta}$ for each of the classifications and seems to indicate that on the average there is some correspondence between the Wiesenberger classifications and our measure of risk. The average $\hat{\beta}$ of the groups declines consistently with our ordering of the classifications, but we note that median values are not nearly as consistent.

[80] This is a truly risk-free rate only in the case of a non-coupon-bearing bond, since in the case of coupon payments this formulation implicitly involves the assumption that all interest payments are reinvested (at time of receipt) at the rate R_F. However, we expect this error to be relatively small, since it ignores only the differential interest earned on the coupon payments, which are themselves quite small.

[81] The $_{10}R^*_M$ is, of course, a ten-year rate of return. Dividing $_{10}R^*_M$ by 10 yields an average annual rate of 11.87 per cent compounded continuously.

[82] As mentioned earlier, we would never expect to see a *perfect* relationship in ex post data, but it is very probable that some of the scatterings of points are due to sampling error in our estimates of β_j as well as the disturbance terms e_j. In practice, this sampling error could be reduced by using monthly or weekly data in the estimation of β_j, but the data are not available in sufficiently convenient form to warrant utilization in this study.

increased risk; and (2) investors' expectations regarding risk were on the average correct and the funds did not substantially alter their risk levels too often.[83]

Thus, there seems to be some empirical as well as theoretical justification for the use of β as the measure of risk. Before turning to the second point mentioned above (the evaluation of fund performance), let us briefly consider the problems associated with the measurement of returns.

The measurement of returns.—Given the assumption that the goal of the investor is to maximize the expected utility of his lifetime consumption pattern, it was shown in Section II that the investor's portfolio choice can be characterized by a single-period utility of terminal wealth model. We reiterate this point since it has implications regarding the relevant measure of returns.

In the context of a single-period utility of terminal wealth model, returns must always be stated in terms of total dollar amounts or (as in this study) a transformation of total returns ($\Delta W/W$) over the *entire* interval of the investor horizon period. We emphasize this point, since most empirical studies utilizing the concepts of risk and return have measured returns as an arithmetic average of *annual* returns[84] which is inconsistent with the underlying utility model. If the distribution of annual returns is skewed, there is no simple and *direct* relationship between an arithmetic average of annual returns and terminal wealth.

For example, assume that the probability distribution of annual returns is log normal with mean $\log_e R = \mu$ and variance σ^2. (Here we are letting $R =$

TABLE 7

AVERAGE AND MEDIAN VALUES OF SYSTEMATIC RISK AND NET RETURN FOR VARIOUS CLASSES OF FUNDS[a]

TYPE OF FUND[b]	NO. OF FUNDS IN CLASS	AVERAGE VALUES		MEDIAN VALUES	
		$\hat{\beta}$	${}_{10}R^*_{1964}$	$\hat{\beta}$	${}_{10}R^*_{1964}$
Growth..........	31	.970	1.018	.919	1.043
Growth-income...	30	.941	1.004	.940	.998
Income-growth...	15	.856	1.037	.834	1.059
Income..........	9	.674	.754	.845	.790
Balanced.........	30	.645	.860	.603	.848

[a] The average and median values of $\hat{\beta}$ for the sample as a whole are given in Table 3. The over-all average return for the sample ${}_{10}\bar{R}^*_{1964} = .955$ and the median value is .961.

[b] As classified by Wiesenberger [67].

[83] If all funds held neutral portfolios and there were no measurement (or sampling) errors in our estimates of the β's, we would expect to find a regression of ${}_{10}R^*_j$ on $\hat{\beta}_j$, yielding coefficients close to those in (6.5), the equation of the market line. We can readily observe that the funds do not all appear to hold neutral portfolios, and we know there are sampling errors in our estimates of the β's. For those interested, however, the estimated regression equation is:

$$ {}_{10}R^*_j = .751 + .243\beta_j \qquad r = .30 $$
$$ (.064) \quad (.074) \qquad n = 115\,. $$

In addition, the reader will note that the weakness of these results is considerably influenced by a few outliers.

[84] Cf. references 7, 27, 31, 54, and 55. Also see n. 78 for a discussion of other problems associated with the use of the arithmetic average of returns.

$1 + r$, where r is the annual return.) While there is a direct monotonic relationship between terminal wealth, W_T, and μ, $W_T = W_t \cdot \exp[\mu \cdot (T - t)]$, there is no such simple relationship between terminal wealth and an arithmetic average of annual returns. Consider the expected value $E(\bar{R})$ of an arithmetic mean $\bar{R}$ calculated from a sample of observations drawn from this log normal distribution. Aitchison and Brown [1, p. 8] show that $E(\bar{R}) = \exp(\mu + \sigma^2/2)$. Thus, the arithmetic mean of a sample from a log normal distribution is a function of the variance of that distribution as well as the mean, μ. Hence, an equal investment in two portfolios having the same arithmetic average of returns over several years will not yield identical values of terminal wealth if their variances differ.

2. *The performance of mutual fund portfolios.*—Since there does seem to be some empirical as well as theoretical justification for the use of β as the measure of risk, we turn to an analysis of the risk-return performance of mutual funds in the period 1945–64. In particular, we address ourselves to the following questions: (1) Have the mutual funds on the average provided investors with returns greater than, less than, or equal to the returns implied by their level of systematic risk and the capital asset pricing model? (2) And have the funds in general provided investors with efficient portfolios?

In attempting to evaluate the funds' performance, particular attention must be given to the treatment of loading charges, management fees, and expenses in calculating fund returns and to the treatment of commission expenses in calculating returns on the market portfolio M. Obviously, in evaluating fund performance from the investor's point of view, the effects of these transaction costs on his returns must be considered,[85] but we defer explicit consideration of these costs for the moment. One can argue that the loading charge (which is generally a pure salesman's commission) actually represents payment for a real economic service, that is, convincing small, uninformed investors of the value of equity investment. Accepting this, the test of a fund management's performance involves only a test of its ability to earn returns sufficiently greater than our naïve *FM* policy to cover the non-loading-charge expenses of the fund—that is, management fees and brokerage expenses. Therefore, the fund returns plotted in Figure 11 were calculated net of all management fees, brokerage commissions, and other expenses incurred by the funds but gross of (ignoring) loading charges to the investors.[86]

In considering the appropriate measure of return on the market portfolio M, one can also argue that the brokerage commissions are analogous to the loading charges and represent payments by the investor for real economic services.

[85] The loading charges can be quite substantial, ranging from zero for the "no load" funds (of which there are thirteen in the sample) to the much more usual charge of 8–8½ per cent of the original investment. The loading charges on the so-called front-end load contractual plans may be substantially higher, often exceeding 30 per cent of the purchase cost if discontinued after two years of their life, 50 per cent if discontinued after one year.

[86] Mutual funds also provide investors with a certain amount of non-monetary returns in the form of bookkeeping services. The investor holding shares in a mutual fund rather than his own diversified portfolio avoids a significant amount of bookkeeping involved in clipping and mailing bond coupons, cashing and recording dividend checks throughout the year, and calculating capital gains and/or losses on any sales executed during the year. Mutual funds generally pay dividends quarterly, and at the end of each year the investor receives a statement detailing the respective amounts he must declare as income and capital gains in his income tax returns. In view of the fact that brokers will also provide similar services, allowance for non-monetary returns has not been incorporated into the analysis.

If we accept this argument for the moment, the appropriate measure of returns on the standard of comparison is the performance of the market portfolio M, ignoring commission expenses. Moreover, while a small investor could not have purchased such a portfolio without incurring high brokerage commissions, most mutual funds could have purchased this portfolio without incurring much larger transaction costs than those incurred in purchasing their actual portfolios. In addition, given that all of our funds started and ended the evaluation period with fully invested portfolios (not cash), it would be inappropriate to charge commissions on the market portfolio alone. Therefore, the point M in Figure 11 represents the risk-return results for the market portfolio calculated without adjustment for commissions.

The scatter of points in Figure 11 gives us a visual impression of the ability of mutual fund managers to choose securities well enough to recoup the expenses they incur in attempting to forecast future prices. The scatter, generally below and to the right of the line ${}_{10}R_F^*MQ$, indicates that in this ten-year period the funds in general provided investors with lower returns than they could have realized by a combined investment in portfolios F and M yielding the same degree of risk. The average value of the difference in returns between the fund portfolio and a comparable FM portfolio, δ_j^* (as defined in eq. [4.22]), is $-.089$, with $\delta_j^* < 0$ for seventy-two funds and $\delta_j^* > 0$ for only forty-three funds. Thus, on the average, these 115 mutual funds earned 8.9 per cent less (compounded continuously) than their comparable FM portfolios over the ten-year period.[87] The performance measures, δ_j^*, for each of the funds are given in column 8 of Table 2, and Figure 12 presents a frequency distribution of the estimates. The funds are ordered from high to low on the basis of δ_j^* in Table 2. We caution the reader to be extremely careful about interpreting these measures without taking into account the sampling errors. Indeed, we shall show below that there is very little evidence that the funds which appear superior were anything more than just lucky in this period. On the other hand, there is evidence that some of the large negative performance measures were not due to chance (see also Jensen [32]).

There is one other matter which warrants our attention at this point. An examination of columns 5 and 8 of Table 2 indicates that the measure of performance δ_j^* is negatively correlated with the measure of risk β_j (the product-moment correlation coefficient between them is $-.68$). This result is not surprising and merely reinforces the hypothesis that much of the variability in the estimates of δ^* is due to random factors or sampling error in the estimates of β_j.

[87] One might also wish to interpret the fund performance in the following manner: Consider a portfolio consisting of an equal dollar investment in each of the fund portfolios and another portfolio consisting of an equal dollar investment in the comparable FM portfolios. One might legitimately ask: How well did the portfolio consisting of investments in the funds perform with respect to the FM portfolio? In order to answer this, we calculate the average terminal wealth ratio of the funds (= 2.639) and take the difference between this and the average terminal wealth ratio of the FM portfolio calculated by

$$(1 + \bar{R}_{FM}) = \frac{1}{115} \sum_{i=1}^{115} \exp\,[R_F^*(1 - \beta_i) + R_M^*\beta_i] = 2.895\,.$$

Thus, the difference, $2.639 - 2.895 = -.256$, indicates that the terminal value of \$1.00 invested in the fund portfolios was 25.6 cents less than the terminal value of \$1.00 invested in the comparable FM portfolios. Or the terminal value of the fund portfolio was $(25.6/2.895) = 8.9$ per cent lower than the FM portfolios. (It should be noted that in general δ will not be equal to the percentage difference in the average wealth ratio, as happens to be the case here.)

To see this more clearly, let us rearrange (4.22) and write it as

$$ {}_{10}R_j^* - {}_{10}R_F^* = \delta_j^* + \beta_j({}_{10}R_M^* - {}_{10}R_F^*) \,, \tag{6.6} $$

giving explicit recognition (by the prescript 10) to the fact that we are considering a ten-year holding period. Now the arguments of Section IV indicate that (6.6) holds for a time interval of any length, so let us consider the annual returns ${}_1R_{jt}^*$ (where the subscript t denotes the year of observation), using (4.22) again to replace δ_j^* with e_j^*, adding an intercept η_j, and interpreting the result

$$ {}_1R_{jt}^* - {}_1R_{Ft}^* = \eta_j + \beta_j({}_1R_{Mt}^* - {}_1R_{Ft}^*) + {}_1e_{jt}^* \tag{6.7} $$

as a regression equation applying to the annual observations over the ten-year period.[88] Now by the additivity of the continuously compounded rates it is clear that

$$ {}_{10}R_j^* = \sum_{t=1}^{10} {}_1R_{jt}^* \,, $$

and likewise for ${}_{10}R_F^*$ and ${}_{10}R_M^*$. Thus, by substitution from (6.7) into (6.6), we have

[88] Equation (6.7) can be used directly for evaluating the performance of portfolios, and it has many convenient properties (the most important of which is the relatively straightforward tests of the significance of the performance measures which it allows). However, since the formulation given by (6.7) is derived and discussed in detail in Jensen [32], we shall not pursue these issues here. We refer the interested reader to that discussion.

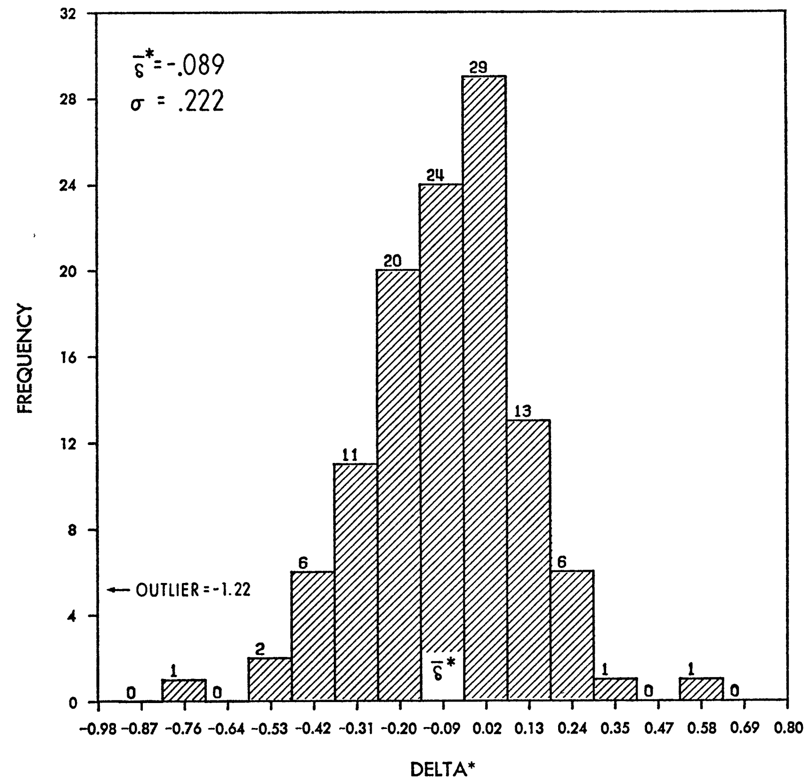

FIG. 12.—Frequency distribution (half-sigma intervals) of the performance measure δ^* for 115 funds for the ten-year period 1955–64.

$$
\begin{aligned}
{}_{10}R_j^* - {}_{10}R_F^* &= \sum_{t=1}^{10}({}_1R_{jt}^* - {}_1R_{Ft}) \\
&= \sum_{t=1}^{10}[\eta_j + \beta_j({}_1R_{Mt}^* - {}_1R_{Ft}) + {}_1e_{jt}] \qquad (6.8) \\
&= 10\cdot\eta_j + \sum_{t=1}^{10} {}_1e_{jt} + \beta_j({}_{10}R_M^* - {}_{10}R_F^*)\,.
\end{aligned}
$$

Thus,

$$\delta_j^* = 10\eta_j + \sum_{t=1}^{10} {}_1e_{jt}^* \qquad (6.9)$$

for the ten-year holding interval under consideration in Table 2. Now, as is clear from (6.9) and as is explained in detail in Jensen [32], η_j is a measure of the forecasting ability of the portfolio manager and has the same properties as δ_j^*. If the manager has no superior knowledge of future security prices, $\eta_j = 0$, and if he has, $\eta_j > 0$.

The reason for deriving (6.9) is to show explicitly that the measure of performance δ_j^* is (except for a scale factor) equal to the intercept η_j plus a sum of random variables, ${}_1e_{jt}^*$, all with zero expected values. Now the effects of sampling error in β_j on the measure of performance δ_j^* become perfectly clear. If we let $\psi_t = {}_1R_{Mt}^* - {}_1R_{Ft}^*$, it is easily shown (cf. Johnston [33, p. 16]) that the correlation between the estimates $\hat{\eta}_j$ and $\hat{\beta}_j$ (and therefore between $\hat{\delta}_j$ and $\hat{\beta}_j$) is given by

$$
\begin{aligned}
r(\hat{\eta}_j, \beta_j) &= r(\delta_j^*, \beta) \qquad (6.10) \\
&= -\bar{\Psi}^2\sigma^2(\beta) \Big/ \left[\frac{\sigma^2({}_1e_j^*)}{n_j} + \bar{\Psi}^2\sigma^2(\beta)\right]^{1/2} \cdot \sigma^2(\beta)\,.
\end{aligned}
$$

In addition, since the average $\sigma^2({}_1e_j^*)$ for our sample of 115 funds is very close to zero (.0032), we see by (6.10) that $r(\hat{\delta}_j^*, \hat{\beta}^*) \cong -1$. Thus, it is not surprising that the estimates of δ^* and β in Table 2 are highly negatively correlated. Of course, the cross-sectional observations are not perfectly correlated, since they are not all generated by the same process; (6.10) applies to repeated samples for a given fund, not a cross-section of different funds.

Returning now to the question of fund performance, the impression gained from an examination of the scatter of points in Figure 11 and the estimates of δ_j^* given in Table 2 is that on the average these mutual funds have not done as well as our very simple and naïve policy of combining an investment in the market portfolio with an investment in government bonds. Now, as long as the capital asset pricing model is valid, the only possible reason for the existence of an inferior portfolio is the unnecessary generation of expenses by the fund managers. These expenses are borne by the fund and therefore reduce the portfolio's returns. In view of this, let us examine the funds in somewhat more detail and give special consideration to these expenses.

The forecasting ability of mutual fund managers.—On the basis of the results considered above, there seems little doubt that the fund managers on the average were unable to predict future security prices well enough to increase returns sufficiently to cover their research and commission expenses. Before reaching a final conclusion regarding the managers' forecasting success, let us consider the possibility of any predictive ability at all—even if insufficient to cover research and transactions costs.

In order to examine this question, let us replicate the analysis of Figure 11 using as our measure of fund returns the total returns gross of *all* expenses except brokerage commissions.[89] That is, let us hypothetically return to the funds all resources spent for security analysis, bookkeeping services, etc., and if the managers have any forecasting ability at all, such ability ought to cause the average δ^* to be positive.

Figure 13 represents the results of the analysis using the gross returns of the funds, and the reader should note that the funds appear to scatter much more equally on either side of the market line ${}_{10}R_F^*MQ$.[90] The average value of δ_j^* calculated on the basis of the gross returns was $-.025$ with fifty-eight funds for which $\delta_j^* < 0$ and fifty-seven for which $\delta_j^* > 0$.[91]

The average δ^* of $-.025$ taken at face value would indicate that on the average the fund portfolios were inferior. But we must recall that these returns were calculated without taking commission expenses into account. Data gathered by Friend *et al.* [26] indicate that the weighted average portfolio turnover rate for mutual funds in the period 1953–58 was about 20 per cent.[92] Adding the brokerage expenses on these transactions (under the assumption that the average commission expense was 1 per cent) would increase the returns by about .002 per year, or about .02 for the ten-year period. This comes very close to accounting for the average δ^* of $-.025$. One other small bias against the funds would account for the remainder of this difference. The standard of comparison, the *FM* portfolios, implicitly assumes a fully invested portfolio, but since the mutual funds face stochastic cash inflows and outflows, they must maintain a cash balance to meet them. On the average, the funds appear to hold about 2 per cent of their total net assets in cash.[93] If we assume that the funds had earned 2.96 per cent on these balances (equivalent to the riskless rate of 3 per cent compounded annually), this would increase their returns by another .0059 for the ten-year period. These adjustments indicate that the actual average δ^* (gross of *all* expenses) is about $+.0009$, which is consistent with the hypothesis that before deduction of expenses the funds held neutral portfolios.

Thus, on the basis of these results (net and gross), we conclude that in the ten-year period 1955–64 mutual fund managers in general showed no evidence of an ability to predict the future performance of securities. That is, they did not as a whole show evidence of superior analytical or forecasting ability in spite of the

[89] It would be desirable to measure the returns gross of brokerage commissions as well as all other expenses, but unfortunately exact commission data is unavailable. We shall consider the effects of these commissions below, using estimates of their average size for a sample of funds.

[90] The gross returns were calculated by adding to the annual net returns the annual expense ratios given by Wiesenberger (÷ 100). The expense ratios are the ratio of total annual expenses, except interest, taxes, and brokerage commissions, to the average total net asset value of the fund (× 100).

[91] The average gross terminal wealth ratio for the funds was 2.813. Thus, the percentage difference in the average terminal wealth ratio for the funds and the comparable risk *FM* portfolios was (2.813 − 2.895)/(2.895 = −.081/2.895 = −.0028 (see n. 87 above).

[92] Actually 19.8 per cent (see Friend *et al.* [26, p. 212]).

[93] Cf. Friend *et al.* [26, pp. 120–27]. The data presented cover four dates in the period 1952–58 and indicate percentages of 2.67 in 1952, 2.03 in 1955 and 1957, and 1.72 in 1958.

considerable resources devoted to these activities.[94, 95]

It is appropriate at this point to remind the reader that these results do not "prove" that security prices behave according to the strong form of the martingale hypothesis given by equation (1.2). They are, however, *consistent* with the joint hypothesis (1) that the capital asset pricing model is valid and (2) that

[94] This evidence should not be construed to imply that there are no particular funds which satisfy the requirements of superior analysts. We consider these questions below.

[95] There is another fragment of evidence bearing directly on the relationship between performance, expenses, and the ability of portfolio managers to predict security prices. Sharpe [55, pp. 132–33], analyzing the performance of thirty-four mutual funds in the period 1954–63, found a negative correlation between fund performance and expense ratios. Our results also tend to support this, but are not nearly as strong as his. The regression of the measure of performance, δ_j^*, on E_j, the average ratio of

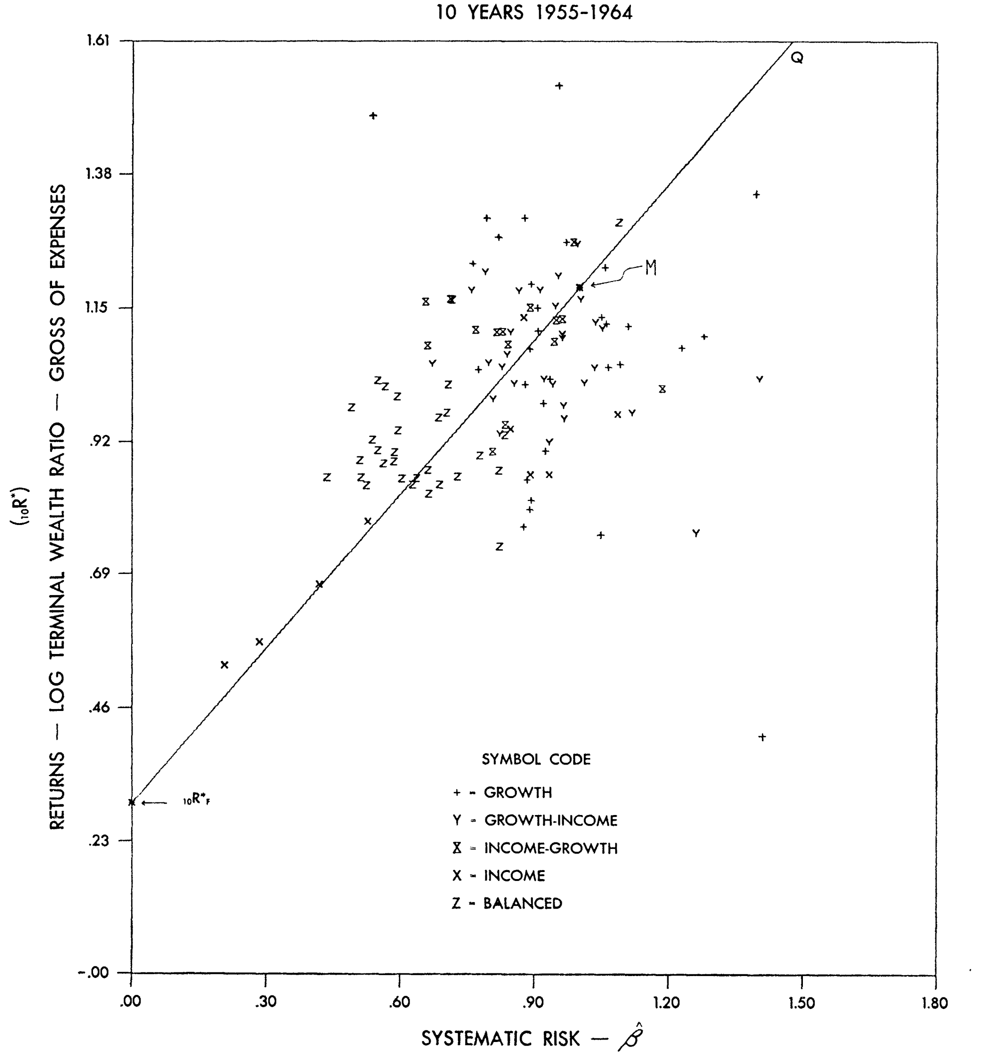

FIG. 13.—Scatter diagram of risk and (gross) return for 115 open-end mutual funds in the period 1955–64

security prices behave according to the strong form of the martingale hypothesis (at least as far as these fund managers are concerned). If the reader is willing to assume that either (1) or (2) above is true, then the results provide a strong piece of evidence in favor of the other hypothesis. Ideally, we should like to test the validity of the capital asset pricing model by using individual assets or unmanaged portfolios. Given the results of these tests, we would then be in a position to make a much stronger statement regarding the apparent validity of the strong form of the martingale hypothesis.

It is well to pause at this point to consider the effects of possible changes in the riskiness of some of the funds. As discussed earlier, the measure of risk, $\hat{\beta}$, is invariant to the length of the time interval over which the returns are measured. Thus, in practice we would prefer to have monthly, weekly, or even daily data over the recent past for use in estimating the present risk of a portfolio. But, given our purposes and the unavailability of such data in convenient form, we have used annual data over the past ten to twenty years. Hence, it is very likely that the risk of some of the funds in our sample may have changed over time, and there is a legitimate question as to the possible effects of this factor on our results.

The analysis of Figure 11 was also performed for the ten-year period 1945–54 on fifty-six of the 115 funds for which data were available. The scatter diagram of net returns and risk for these funds is given in Figure 14.[96] The analysis utilizing gross returns could not be replicated for this period, since sufficient expense data were not available. For comparison purposes, the results for these fifty-six funds in the period 1955–64 appear in Figure 15.

Figure 14 seems to imply that the measure of systematic risk is also appropriate for this earlier period. The scatter indicates a positive relationship between risk and return, as the theory of capital asset prices implies.[97]

It appears that after expenses the funds' returns were much farther below the returns on possible FM combinations in the period 1945–54 than the later period 1955–64. The average δ^* for these fifty-six funds was $-.135$ in the earlier

all expenses (except interest, taxes, and commissions) to net assets in the period 1955–64, yields:

$$\delta_j^* = .026 - .164E_j \qquad r = .175$$
$$(.075) \quad (.102) \qquad n = 115 ,$$

where the δ_j^* are measured net of expenses.

These results, although weak, are at least *consistent* with the hypothesis that mutual fund managers as a group cannot forecast prices any better than the average investor in the market. Hence, the more resources they devote to forecasting and the more commission expenses they incur in implementing the trading advice of their research departments, the smaller will be their ex post returns. We note that the opposing hypothesis (i.e., that prices can be predicted and profits increased by buying good advice) implies a *positive* relationship between δ_j^* and E_j as long as funds devote resources to research and trading only to the point where expected marginal revenue equals the marginal cost.

96 The risk-free rate of interest in this period was taken to be 2.1 per cent (the ten-year yield to maturity on government bonds in 1945) resulting in a ten-year riskless return, ${}_{10}R_F^* = \log_e (1 + {}_{10}R_F) = \log_e (1.230) = .208$. The return on the market portfolio, ${}_{10}R_M^*$, was 1.536 over this period. The equation of the market line, ${}_{10}R_F^*MQ$, is ${}_{10}R^* = .208 + 1.328\,\hat{\beta}$.

97 Although again we wish to place no interpretation on the estimated regression of ${}_{10}R_j^*$ on $\hat{\beta}_j$, we present the results here for whatever interpretation the reader may desire to make for himself:

$$ {}_{10}R_j^* = .477 + .841\beta_j \qquad r = .67$$
$$(.109) \quad (.128) \qquad n = 56 .$$

It should be noted that the outlier in the upper right corner of Figure 14 (the Investment Trust of Boston again) is the major reason for the much higher correlation in this case as compared with the results shown in note 83 above for the period 1955–64. (Also see n. 75 above for a discussion of this fund.)

period and $-.076$ in the later period, with forty-three and thirty-five funds having $\delta_j^* < 0$, respectively, in the two periods.[98]

The analysis was also performed for the entire twenty-year period 1945–64, and the scatter of risk versus net return for the fifty-six funds for which data were available is given in Figure 16. The risk-free rate was taken to be 2.1 per cent,[99] giving ${}_{20}R_F^* = \log_e\ (1.021)^{20} =$

[98] The average terminal wealth ratio for these fifty-six funds over the period 1945–54 was 3.347 and that for the comparable risk *FM* portfolios (see n. 87 above) was 3.838. Thus, the percentage difference in the average terminal wealth ratios was $(3.347 - 3.838)/3.838 = -.128$. Hence, the returns on an equal dollar investment in the shares of these fifty-six funds in the period 1945–54 was 12.8 per cent less than the returns which could have been obtained by holding a comparable risk *FM* portfolio.

[99] The yield to maturity of a twenty-year government bond as of January, 1945.

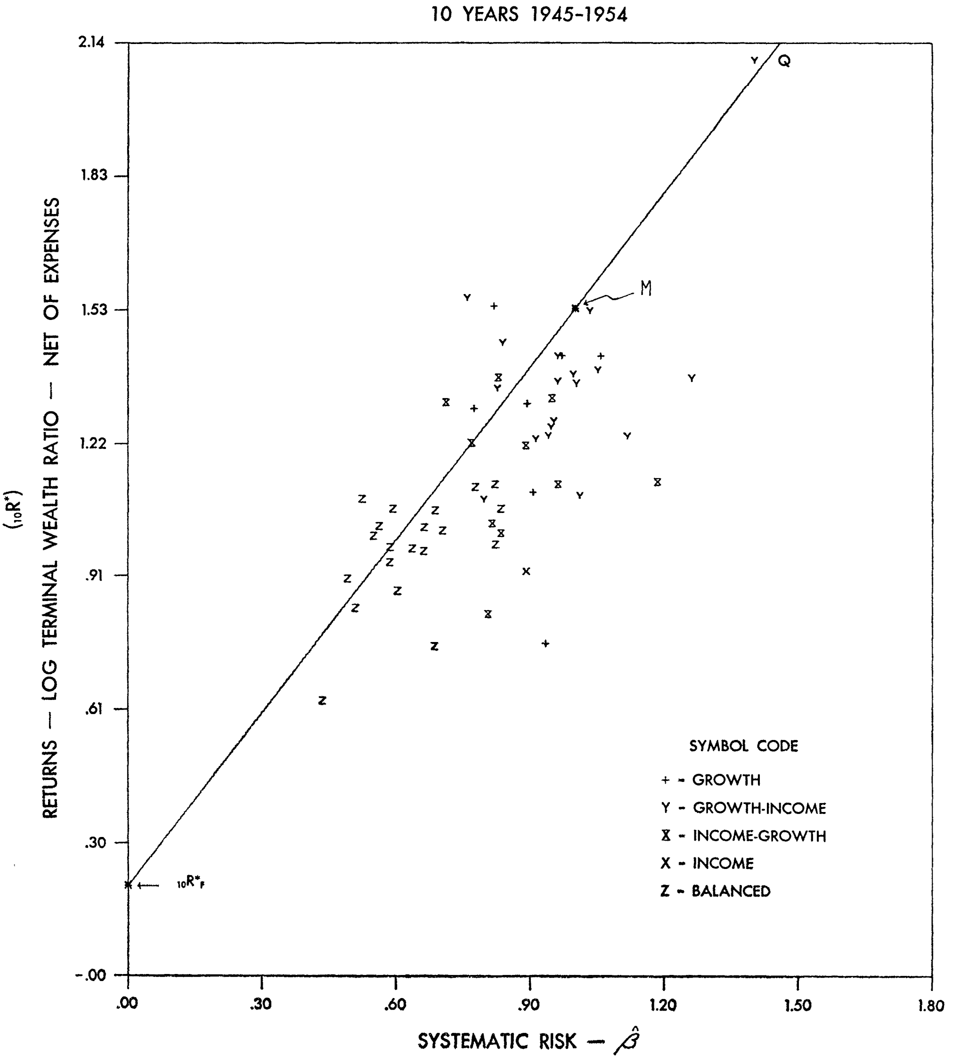

FIG. 14.—Scatter diagram of risk and (net) return for fifty-six open-end mutual funds in the ten-year period 1945–54.

.416. The return on the market portfolio $_{20}R_M^*$ was $\log_e$ (15.23) = 2.723, or an annual return of 13.61 per cent compounded continuously. Thus, the equation of the market line is $_{20}R^* = .416 + 2.307\,\hat{\beta}$.

The average δ^* for this twenty-year period was −.196, with thirty-nine funds for which $\delta_j^* < 0$ and seventeen for which $\delta_j^* > 0$.[100, 101]

These results are not surprising in light of the previous results. A more interesting facet of the scatter of Figure 16 is its apparent linearity. The reader will

[100] Again we present the regression results of $_{20}R_j^*$ on β_j for the interested reader. The results are:

$$_{20}R_j^* = 1.194 + 1.133\beta_j \qquad r = .63$$
$$(.162) \quad (.190) \qquad n = 56\,.$$

[101] The average terminal wealth ratio for these fifty-six funds over the twenty-year period 1945–64

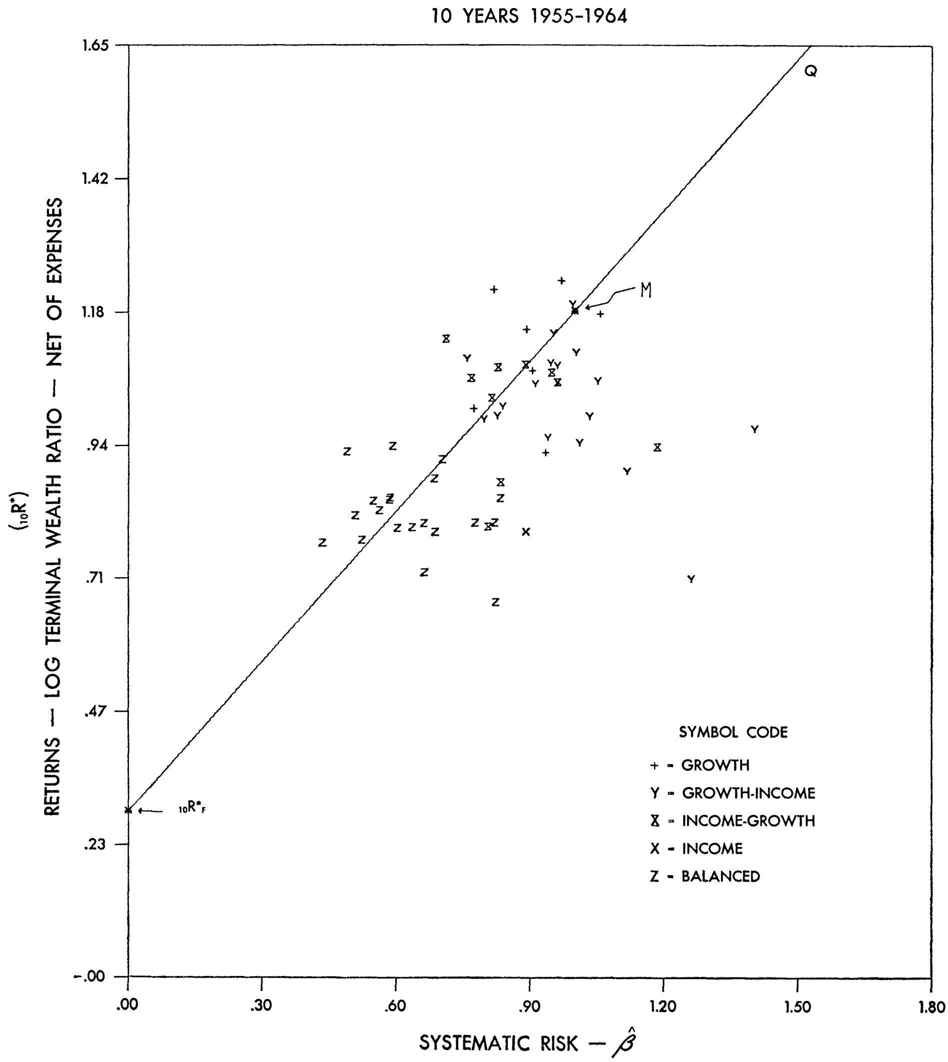

FIG. 15.—Scatter diagram of risk and (net) return over the ten-year period 1955–64 for the fifty-six mutual funds existing in the period 1945–54.

remember that it was argued in Section IV that the linear opportunity set will hold only for the variable R^* defined as $[(1 + R)^{H/N} - 1]/(H/N)$. Several arguments were also presented which lead us to believe that H, the "market horizon" interval, is very close to zero. It was also shown that for H/N close to zero, the variable R^* will be very well approximated by $\log_e (1 + R)$. Hence, the apparent linearity of the scatter of Figure 16 is another piece of evidence supporting our arguments regarding the length

was 9.01, and that for the comparable risk *FM* portfolios (see n. 87 above) was 11.45. Thus, the percentage difference in the average terminal wealth ratios was (9.01 − 11.45)/11.45 = −.27, and the returns on an equal dollar investment in the shares of these funds in this twenty-year period were 27 per cent less than the returns which could have been obtained by holding a comparable risk *FM* portfolio.

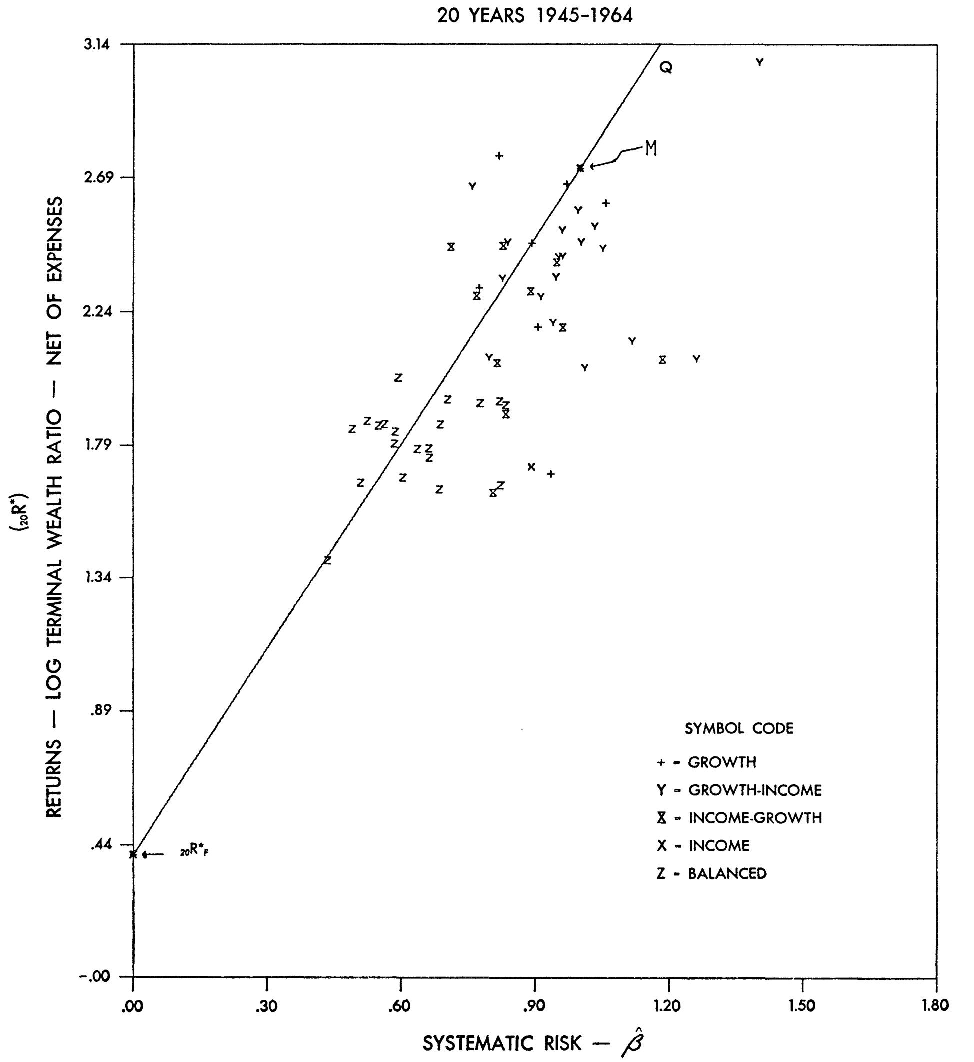

FIG. 16.—Scatter diagram of risk and (net) return over the 20-year period 1945-64 for fifty-six mutual funds.

of the "market horizon" interval. However, due to the presence of substantial measurement errors in both ${}_{20}R_j^*$ and $\hat{\beta}_j$, we are unwilling to place great emphasis on these arguments at this time.[102]

The twenty-year scatter of risk versus returns is plotted in *arithmetic* form in Figure 17. That is, the returns $(1 + {}_{20}R)$ plotted on the abscissa are the twenty-year wealth relatives $W_{20}/W_0 = \exp$

[102] These measurement errors arise primarily from sampling errors in the estimation of $\hat{\beta}$ and the inability to measure true gross returns of the funds (i.e., before deduction of brokerage commissions and management expenses). Thus, the errors can be reduced, and work in progress on these problems at this time should allow us to obtain much better tests of the adequacy of the capital asset pricing model and the horizon solution.

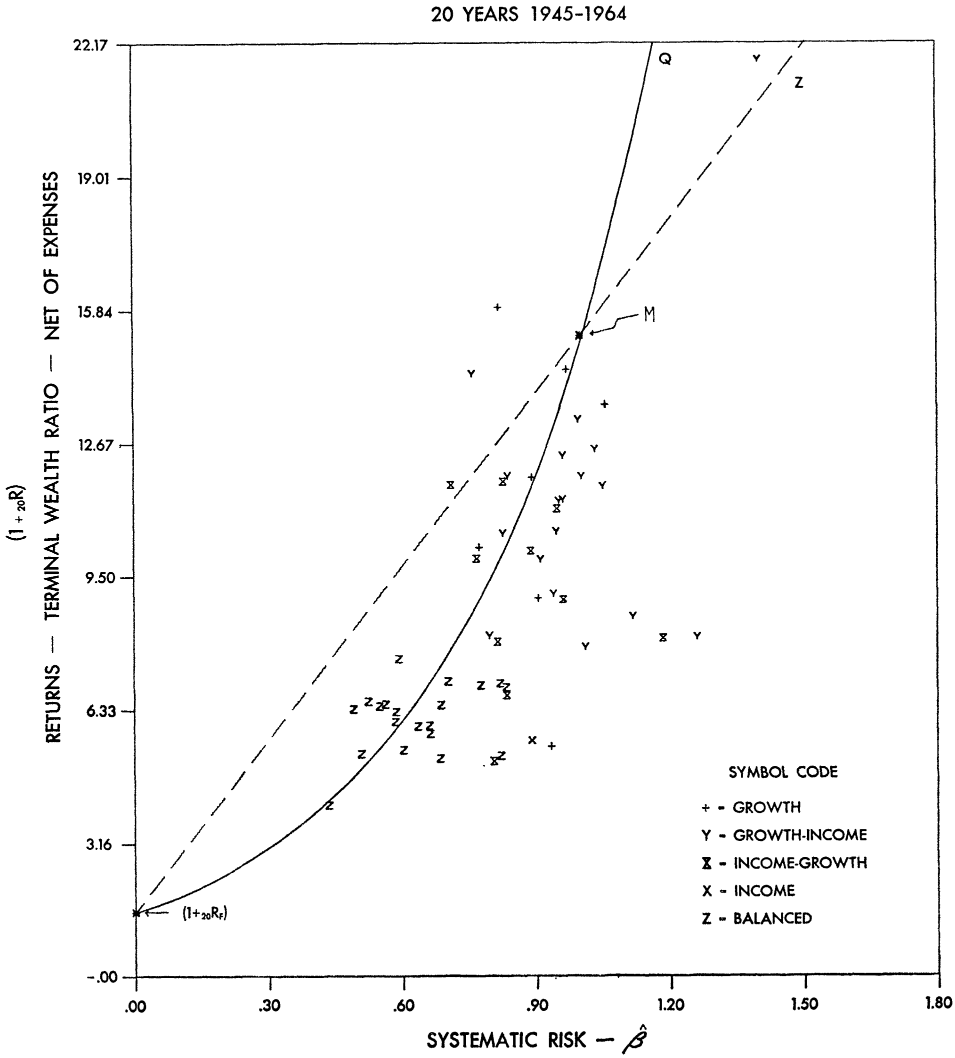

FIG. 17 —Scatter diagram of risk versus arithmetic return (wealth relatives) for fifty-six funds over the twenty-year period 1955–64. Dashed line represents the "market line" which would be given by a naïve interpretation of the capital asset pricing model.

($_{20}R^*$). There is a suggestion of nonlinearity. The scatter certainly does not conform well to the dashed market line $(1 + {}_{20}R_F)MZ$ expressed as

$$(1 + {}_{20}R) = 1.52 + (15.23 - 1.52)\hat{\beta}$$
$$= 1.52 + 13.71\hat{\beta} \,,$$

which would be given by a naïve interpretation of the capital asset pricing model. On the other hand, they do seem to conform more closely[103] to the solid line $(1 + {}_{20}R_F)MQ$, which is given by

$$(1 + {}_{20}R) = \exp\,[R_F^* + (R_M^* - R_F^*)\hat{\beta}]$$
$$= \exp\,(.416 + 2.307\hat{\beta}) \,,$$

which is the inverse of the limit of (4.13). It might also be recalled from the discussion in Section IV-B that this curve is the limiting form of the market line as H, the "market horizon," goes to zero. Thus, if the arguments given in Section IV-B regarding the length of the market horizon are erroneous and the "true" market horizon is actually significantly greater than zero (but less than twenty years), then the "true" market line (given by the inverse of the function defined in [4.10] applied to [4.11]) will lie somewhere between the line $(1 + {}_{20}R_F)MZ$ and $(1 + {}_{20}R_F)MQ$ passing through the points $(1 + {}_{20}R_F)$ and M. (But note that a "true" horizon interval of H equal to one month will not yield a market line which is visibly different from $[1 + {}_{20}R_F]MQ$.)

Fund performance from the investor's point of view.—An evaluation of mutual fund performance from the investor's point of view must allow explicitly for the effects of transaction costs and loading fees on returns in going from cash to portfolio and back to cash at the horizon date. Thus, in calculating the net returns to the investor for this analysis, explicit allowance was made for the actual loading charge for each fund[104] in 1955 as well as all other expenses incurred by the fund. The brokerage commissions on the purchase and sale of the market portfolio were assumed to be 1 per cent each way.

The average δ^* calculated under these assumptions for the ten-year period 1955–64 was $-.146$, with $\delta^* < 0$ for eighty-nine of the 115 funds. The scatter of points is given in Figure 18.

A question of consistency through time.—We observed earlier that we expect the ex post results of neutral portfolios to be randomly distributed about the market line. We have also seen that the funds in our sample *on the average* held somewhat inferior portfolios in both periods. It also appears that some funds hold neutral portfolios, and we might legitimately ask the question: Do individual funds consistently hold either superior or inferior portfolios? One way to address this question is to examine the relationship between the performance measure,[105] δ_j^*, for each fund in the two ten-year periods previously analyzed. The regression of $\delta^*_{j,1955-64}$ on $\delta^*_{j,1945-54}$ for the fifty-six funds observed in both periods yields the following results:

$$\delta^*_{j,1955\text{-}64} = \underset{(.022)}{-.003} + \underset{(.089)}{.542}\delta^*_{j,1945\text{-}54} \quad r = .64 \,.$$

Thus, for this sample of fifty-six funds, positive correlation exists between the performance measures in the two periods, indicating that some funds may be con-

[103] But again, the existence of measurement errors prevents the formulation of firm conclusions on this point.

[104] That is, the ten-year returns net of transactions costs were calculated as $\log_e x_j(1 + {}_{10}R_j)$, where x_j is defined as the ratio of the net asset value to the offering price per share of the jth fund and ${}_{10}R_j$ is the return net of all expenses for the jth fund.

[105] Calculated net of all management expenses and brokerage commissions but gross of (excluding) loading charges.

sistently inferior and others consistently superior. However, we must be very careful in interpreting these results to mean that a fund manager who experienced *superior* performance in the earlier period was far more likely to experience superior results in the latter period. In fact, an examination of Figure 19 indicates that the correlation of +.64 is mainly due to the points in the third quadrant. That is, a fund which was *inferior* in the earlier period was very likely to be inferior in the latter period. This result is not too surprising, since it is very simple to consistently hold an inferior portfolio. In the absence of forecasting ability, all one need do is generate substantial expenses through time to insure inferior performance.

In order to test this consistency ques-

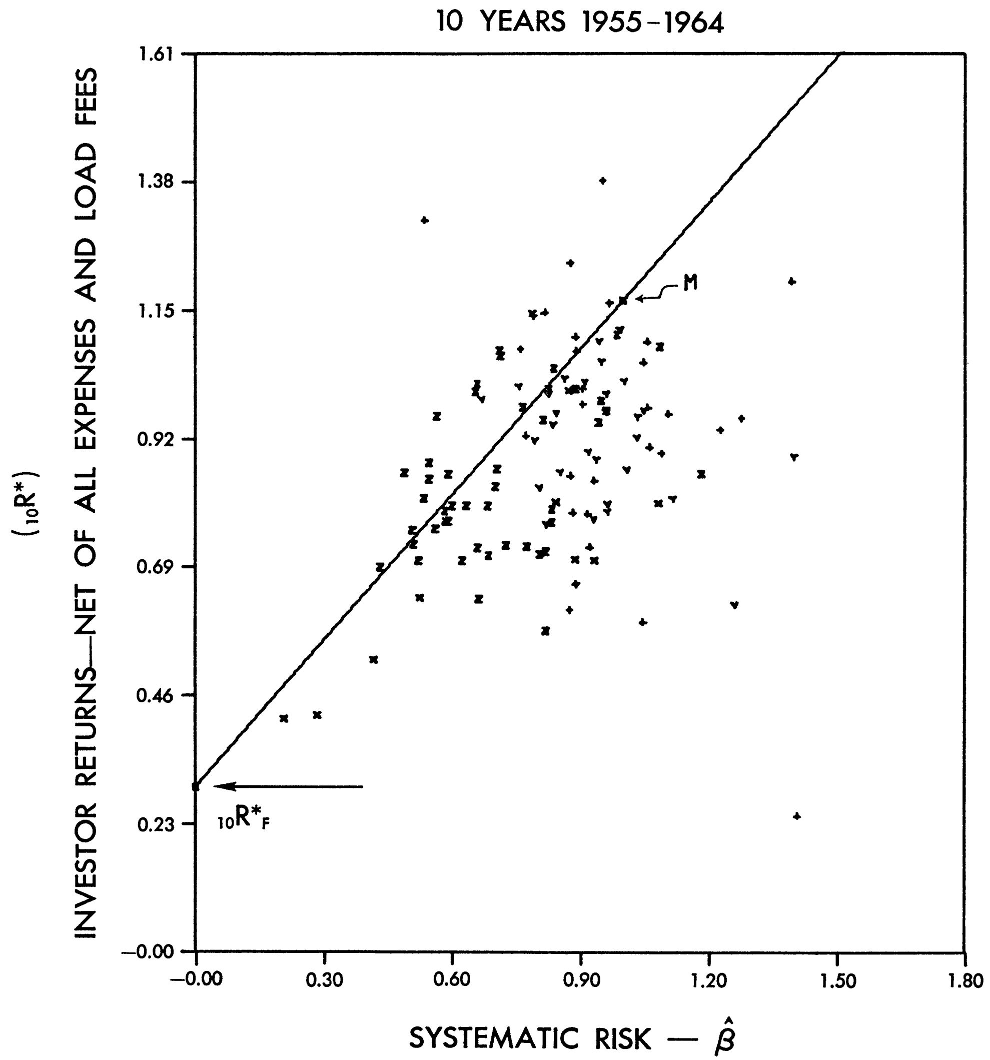

FIG. 18.—Scatter diagram of risk and returns (measured net of all expenses including load fees) for 115 mutual funds in the ten-year period 1955–64.

tion more fully, the analysis was repeated for yearly holding periods and the δ_j^* were calculated for each fund for all years in which return data were available in the period 1945–64. Table 8 presents a summary of these results for each fund. The funds are ranked in ascending order on the basis of the fraction of the number of observations for each fund for which $\delta_j^* > 0$. The δ_j^* were calculated on the basis of returns gross of expenses (except interest, taxes, and commissions) for the ten years 1955–64 and net of expenses for the ten years 1945–54, since expense data were unavailable for the earlier period. The general impression gained from an examination of Table 8 is that most funds (sixty-nine out of 115) had more negative than positive performance measures. Furthermore, no fund experienced positive δ_{jt}^*'s for more than 80 per cent of its total number of observations.

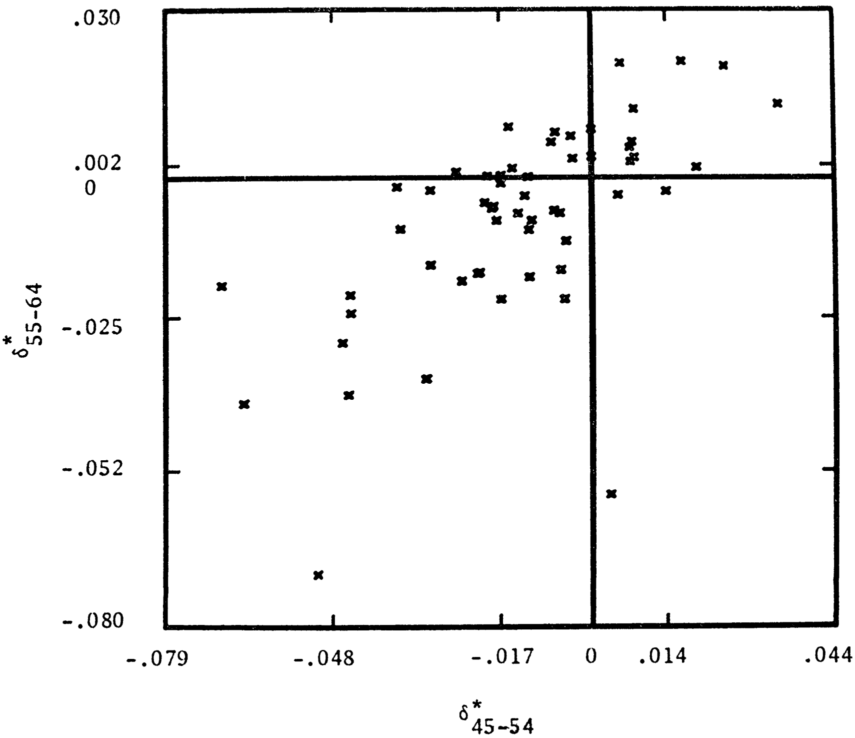

FIG. 19.—Scatter diagram of $\delta^*_{1955-64}$ versus $\delta^*_{1945-54}$ showing the consistency of fund performance in the two ten-year periods for the fifty-six funds having twenty years of complete data.

We have seen that net of expenses, the fund portfolios seem to be inferior. Thus, in order to test for the existence of any consistency, let us focus our attention somewhat more closely on the δ^{*}'s calculated gross of expenses during the ten-year period 1955–64. The total number of observations in this ten-year period for the 115 funds was 1,150, of which 578 (50.2 per cent) were positive and 572 were negative. Again we find no evidence of superior ability for the sample as a whole.

Now let us examine the conditional probability of observing a positive δ_j^* for

TABLE 8

FUND RANKS BASED ON THE PERFORMANCE MEASURE, δ_j, FOR ANNUAL HOLDING PERIODS IN THE INTERVAL 1945–64

	PERCENT +	NUM +	NUM OBS	ID NUM
1	0.050	1.000	20.000	198
2	0.050	1.000	20.000	222
3	0.091	1.000	11.000	146
4	0.100	2.000	20.000	220
5	0.100	2.000	20.000	2223
6	0.118	2.000	17.000	160
7	0.143	2.000	14.000	2209
8	0.150	3.000	20.000	184
9	0.150	3.000	20.000	147
10	0.167	2.000	12.000	155
11	0.167	2.000	12.000	254
12	0.182	2.000	11.000	167
13	0.200	4.000	20.000	171
14	0.200	4.000	20.000	187
15	0.214	3.000	14.000	2211
16	0.231	3.000	13.000	210
17	0.231	3.000	13.000	158
18	0.231	3.000	13.000	164
19	0.250	5.000	20.000	205
20	0.250	5.000	20.000	145
21	0.250	5.000	20.000	2148
22	0.250	5.000	20.000	243
23	0.250	5.000	20.000	153
24	0.286	4.000	14.000	253
25	0.286	4.000	14.000	231
26	0.300	6.000	20.000	140
27	0.300	6.000	20.000	174
28	0.300	3.000	10.000	197
29	0.308	4.000	13.000	1191
30	0.308	4.000	13.000	224
31	0.308	4.000	13.000	194
32	0.333	6.000	18.000	1261
33	0.350	7.000	20.000	166
34	0.350	7.000	20.000	180
35	0.350	7.000	20.000	240
36	0.350	7.000	20.000	217
37	0.350	7.000	20.000	188
38	0.350	7.000	20.000	182
39	0.350	7.000	20.000	2191
40	0.357	5.000	14.000	1209
41	0.364	4.000	11.000	1159
42	0.368	7.000	19.000	202
43	0.389	7.000	18.000	189
44	0.389	7.000	18.000	1223
45	0.400	8.000	20.000	216
46	0.400	8.000	20.000	185
47	0.400	8.000	20.000	236
48	0.400	8.000	20.000	219
49	0.400	8.000	20.000	195
50	0.400	8.000	20.000	245
51	0.400	4.000	10.000	247
52	0.400	8.000	20.000	251
53	0.400	8.000	20.000	190
54	0.400	6.000	15.000	168
55	0.400	8.000	20.000	259
56	0.400	8.000	20.000	260
57	0.400	8.000	20.000	1148
58	0.400	8.000	20.000	263
59	0.429	6.000	14.000	1211
60	0.429	6.000	14.000	255
61	0.437	7.000	16.000	193
62	0.437	7.000	16.000	249
63	0.450	9.000	20.000	252
64	0.450	9.000	20.000	241
65	0.450	9.000	20.000	178
66	0.450	9.000	20.000	2261
67	0.450	9.000	20.000	141
68	0.462	6.000	13.000	150
69	0.471	8.000	17.000	235
70	0.500	8.000	16.000	244
71	0.500	7.000	14.000	192
72	0.500	10.000	20.000	221
73	0.500	10.000	20.000	142
74	0.500	10.000	20.000	163
75	0.500	10.000	20.000	157
76	0.500	5.000	10.000	165
77	0.500	7.000	14.000	172
78	0.500	7.000	14.000	2212
79	0.500	6.000	12.000	256
80	0.500	10.000	20.000	215
81	0.500	5.000	10.000	239
82	0.500	10.000	20.000	200
83	0.500	10.000	20.000	201
84	0.500	6.000	12.000	173
85	0.529	9.000	17.000	207
86	0.538	7.000	13.000	2159
87	0.550	11.000	20.000	257
88	0.550	11.000	20.000	151
89	0.562	9.000	16.000	218
90	0.571	8.000	14.000	227
91	0.571	8.000	14.000	208
92	0.571	8.000	14.000	144
93	0.571	8.000	14.000	1212
94	0.579	11.000	19.000	203
95	0.583	7.000	12.000	1000
96	0.588	10.000	17.000	176
97	0.600	12.000	20.000	152
98	0.600	12.000	20.000	233
99	0.600	12.000	20.000	225
100	0.600	12.000	20.000	226
101	0.600	12.000	20.000	175
102	0.632	12.000	19.000	1268
103	0.643	9.000	14.000	2186
104	0.643	9.000	14.000	234
105	0.650	13.000	20.000	177
106	0.667	10.000	15.000	232
107	0.667	10.000	15.000	250
108	0.700	14.000	20.000	162
109	0.727	8.000	11.000	206
110	0.733	11.000	15.000	2262
111	0.733	11.000	15.000	246
112	0.765	13.000	17.000	2268
113	0.786	11.000	14.000	1186
114	0.800	8.000	10.000	267
115	0.800	16.000	20.000	169

the jth fund, given that the fund has experienced a run of k positive δ^*'s in the immediately preceding k years. Table 9 presents the results for runs of k from 1 to 7.[106] None of these frequencies are significantly different from what would be expected under the assumption of independence. Hence, again there is no striking evidence of superior forecasting ability on the part of the funds in the sample.

Finally, we note that the somewhat alternative formulation of the evaluation model given by equation (6.7) allows us to perform formal tests of significance on the superiority of a portfolio. This formulation is developed in detail in Jensen [32] and need not be repeated here. Suffice it to say that the analysis in Jensen [32] was applied to the same sample of 115 funds as used here, and the results indicate very little evidence that any *individual* fund was able to earn significantly higher returns than those which could have been expected merely from random choice.

The efficiency of mutual fund portfolios.—Up to this point, we have been concerned primarily with the forecasting ability of the mutual fund managers and have concerned ourselves only with the question of neutrality. Recalling the discussion of efficiency in Section V, it is clear that we are now in a position to draw some conclusions regarding the efficiency of the portfolios in our sample.

First of all, we know that since the average correlation coefficient for the sample as a whole is .923 (cf. Table 3), the funds in general hold very well-diversified portfolios. (Recall that a perfectly diversified portfolio will have a correlation coefficient of unity.) Perhaps the implications of this fact are more easily seen by comparing the total relative risk $[\sigma(R_j)/\sigma(R_M) = (1/r_j)\beta_j]$ of the portfolios given in column 6 of Table 2 with the systematic risk $\hat{\beta}_j$ given in column 5. Figure 20 presents a scatter diagram of the estimated total relative risk of the portfolios $\hat{\beta}_j/\hat{r}_j$ against the systematic risk $\hat{\beta}_j$. A 45-degree line represents the points upon which perfectly diversified portfolios would lie. The scatter in Figure 20 and the fact that the average total relative risk is only .908 as com-

TABLE 9

CONDITIONAL FREQUENCY OF POSITIVE δ^*'S FOR 115 FUNDS IN THE PERIOD 1955–64

k—Length of Run of Positive δ^*'s Prior to Year t (1)	No. of Occurrences (2)	No. of Positive δ^*'s in Year t (3)	No. of Negative δ^*'s in Year t (4)	Relative Frequency of Positive δ^*'s (5) = (3)/(2)
1..........	574	289	285	.504
2..........	312	162	150	.520
3..........	161	86	75	.534
4..........	79	44	35	.558
5..........	41	19	22	464
6..........	17	6	11	.353
7..........	4	1	3	.250

[106] I am indebted to Fischer Black of Arthur D. Little, Inc., for suggesting this test and for permission to quote the results given in Table 7, which were calculated by him on the fund performance data supplied by me.

pared with the average $\hat{\beta}$ of .840 indicate that in general the funds held very well-diversified portfolios.

Recall now the relationship between the measure of efficiency, γ_j^*, and the measure of performance, δ^*, given by (5.21):

$$\gamma_j^* = \delta_j^* - [E(R_M^*) - R_F^*]\beta_j \times \left(\frac{1}{r_j} - 1\right) \qquad r_j \neq 0 . \tag{5.21}$$

If $r_j = 1$, we know that $\gamma_j^* = \delta_j^*$, and for all portfolios for which $r_j < 1$, we know that γ_j^* must be less than δ_j^*. Thus, it is clear that after deducting expenses, the funds in our sample in general held portfolios which meet our definition of inefficiency (since the average δ^* for the period 1955–64 was $-.089$). The average γ^* must be less than $-.089$ for this period.

For illustrative purposes, we have calculated measures of efficiency γ_j^* under the assumption that $E(R_M^*)$ for the period 1955–64 was equal to the realized returns, R_M^*, of 1.187 (11.87 per cent per year compounded continuously). The estimates for the individual funds[107] are given in column 9 of Table 2 and are summarized in a frequency distribution in Figure 21.

The average γ^* is $-.150$ with a minimum of -1.447 and a maximum of .427. In addition, there are twenty-nine funds for which $\gamma_j^* > 0$ and eighty-six funds for which $\gamma_j^* < 0$. Thus, under the assumption that $E(R_M^*) = 1.187$ for the period 1955–64, we see that the mutual

[107] These estimates are, of course, dependent upon the assumption regarding the value of $E(R_M^*)$ for the period 1955–64. The reader who does not like the assumption that $E(R_M^*) = R_M^*$ over this period can readily calculate his own measures given the data in Table 2.

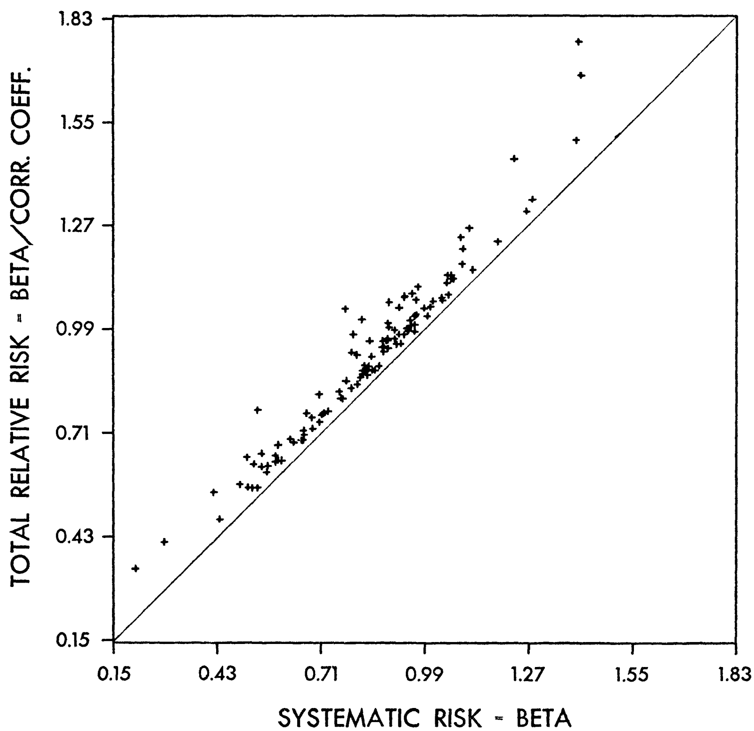

FIG. 20.—Scatter diagram of total relative risk $[\sigma(R_j^*)/\sigma(R_M) = \beta_j/r_j$ and systematic risk β_j for 115 mutual funds. Note that the 45-degree line represents perfect diversification.

funds in our sample on the average earned 15 per cent less per year than an efficient portfolio would have earned.[108]

VII. SUMMARY AND CONCLUSIONS

A. SUMMARY OF THE MAIN RESULTS

The theoretical results.—It was shown in Sections II and III that the Sharpe-Lintner theory of capital asset pricing can be used to develop a model for evaluating the performance of portfolios. The model uses the Sharpe-Lintner results to allow explicitly for the effects of differential degrees of "risk" on the returns of portfolios—a problem which prior to this time has never been satisfactorily solved.

The Sharpe-Lintner results (originally derived in the context of a single-period model under the assumption of identical investor horizon periods) were extended to a multiperiod world in Section IV. In this model, investor horizon periods may be of different lengths and trading of assets is allowed to take place continuously.

In addition, the Sharpe-Lintner ex ante model was extended to include ex post relationships. That is, the resulting model expresses the expected returns on a security (or portfolio) as a function of its level of systematic risk, the risk-free return, and the *actual realized returns* (in-

[108] We note at this point that one might wish to use the ratio of the calculated $\sigma(R_j^*)$ and $\sigma(R_M^*)$ over the actual sample interval for the estimate of total relative risk rather than $\hat{\beta}/\hat{r}_j$. We use the latter here for the same reasons that we used all available data in calculating $\hat{\beta}$. Unless one has reason to believe that the process is non-stationary, this procedure will tend to minimize the sampling error in the estimates.

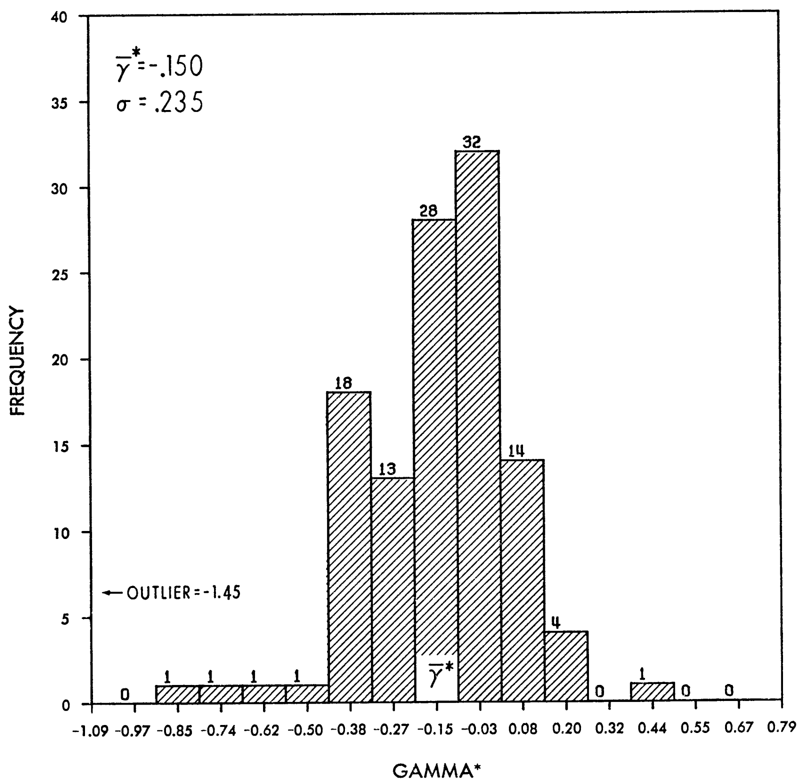

FIG. 21.—Frequency distribution (half-sigma intervals) of the measure of efficiency γ^* for 115 funds for the period 1955–64 (calculated under the assumption that $E[R_M^*] = R_M^*$ for the period).

stead of the expected future return), on the market portfolio over any holding period.

Given these results, a measure of portfolio performance was defined as the difference between the actual returns on a portfolio in any particular holding period and the expected returns on that portfolio conditional on the riskless rate, its level of systematic risk, and the actual returns on the market portfolio. The criteria for judging portfolio performance to be neutral, superior, or inferior were established in Section V-A.

The concept of efficiency was explicitly defined in Section V-B and a measure of efficiency was derived. It was also shown that it is strictly impossible to define a measure of efficiency solely in terms of ex post observable variables. In addition, it was shown that there exists a natural relationship between the measure of portfolio performance and the measure of efficiency.

The empirical results.—The empirical tests presented in Section VI yielded the following results:

1) As implied by the solution to the horizon problem, the estimates of systematic risk seem to be invariant to the length of the interval over which the sample returns are calculated. If this conclusion is valid, the estimates of systematic risk are independent of the investor's horizon period and may be used to evaluate portfolios over a holding period of any length.

2) The measures of systematic risk for the mutual funds seem to be approximately stationary over time, implying that we may use historical data on returns to estimate a portfolio's level of risk.

3) The observed historical patterns of systematic risk and return for the mutual funds in the sample are consistent with the joint hypothesis that the capital asset pricing model is valid and that the mutual fund managers on the average are unable to forecast future security prices.

4) If we assume that the capital asset pricing model is valid, then the empirical estimates of fund performance (summarized in Table 10) indicate that the fund portfolios were "inferior" after deduction of all management expenses and brokerage commissions generated in trading activity. Under these conditions, the average performance measure, $\bar{\delta}^*$, for the 115 funds was -8.9 per cent in the period 1955–64. In addition, when all management expenses and brokerage commissions were added back to the fund returns and the average cash balances of the funds were assumed to earn the riskless rate, the fund portfolios appeared to be just neutral. The average δ^* was $+.0009$ in the period 1955–64. Thus, it appears that on the average the resources spent by the funds in attempting to forecast security prices do not yield higher portfolio returns than those which could have been earned by equivalent risk portfolios selected (*a*) by random selection policies or (*b*) by combined investments in the market portfolio and government bonds.

5) Based on the evidence summarized above, we conclude that as far as these 115 mutual funds are concerned, prices of securities seem to behave according to the "strong" form of the martingale hypothesis outlined in the Introduction. That is, it appears that the current prices of securities completely capture the effects of *all* currently available information. Therefore, their attempts to analyze past information more thoroughly have not resulted in increased returns.

6) Given the results regarding the average performance measure, we also conclude that on the average the mutual funds provided investors with inefficient

portfolios. The explicit estimates of a measure of efficiency yielded an average γ^* of $-.150$ under the assumption $E(R_M^*) = R_M^*$ for the period 1955–64.

7) The evidence also indicates that, while the portfolios of the funds on the average are inferior and inefficient, this is due mainly to the generation of too many expenses. We know that, since the portfolios on the average are very well-diversified (with an average r_j of .923), they are inefficient mainly because they are inferior, and this apparently is a result of the generation of too many expenses. That is, after adding back all expenses except brokerage commissions and adjusting for the bias involved with the cash balances, the portfolios satisfy the criterion for neutrality with an average δ^* of +.0009.

TABLE 10

SUMMARY OF FUND PERFORMANCE MEASURES (δ^*) BY TIME PERIOD AND ASSUMPTIONS REGARDING TREATMENT OF EXPENSES

Definition of Sample and Treatment of Expenses and Transaction Costs	Sample Size	Ten Years 1955–64 unless Noted Otherwise		
		Average δ^*	Funds with Negative δ^*	
			No.	%
(1) Total sample—fund returns calculated after subtraction of all expenses; transaction costs ignored....	115	−.089	72	62.5
(2) Fifty-six funds existing over entire 20 years; returns calculated as in (1) above..............	56	−.076	35	62.5
ten years 1945–54..........................	56	−.135	43	76.8
twenty years 1945–64........................	56	−.196	39	69.5
(3) Total sample—fund returns calculated after adding back all expenses except interest, taxes, and brokerage commissions; transaction costs on market portfolio ignored..............................	115	−.025	58	50.4
(4) Total sample—fund returns calculated gross of all reported expenses and estimated commission expenses; estimated average cash balances assumed to earn the riskless rate; transaction costs on market portfolio ignored...........................	115	+.0009	N.A.[a]	N.A.
(5) Total sample—fund and market portfolio returns calculated as seen by a potential investor; all fund expenses, brokerage commissions, and loading fees subtracted and brokerage commissions of 1% on purchase and sales of market portfolio allowed for..	115	−.146	89	77.4

[a] N.A. = insufficient data available to make calculations.

Implications of the results for mutual fund investment policies.—The results of the analysis imply that in the absence of superior forecasting ability, mutual funds ought to maintain the following policies in order to provide investors with maximum benefits:

1) Minimize management expenses and brokerage commissions. That is, a buy-and-hold policy should be followed as closely as possible.

2) Concentrate on the maintenance of a perfectly diversified portfolio.

In addition to the above implications (which are direct results of the analysis), considerations of the utility model and the manner in which anticipations regarding future risk are formed imply that mutual fund managers should also:

3) Maintain a constant level of sys-

tematic risk as closely as possible. A fund which establishes a risk level and attracts investors on this basis should avoid sudden shifts in its risk level, since unexpected changes in its risk are likely to leave its investors with inappropriate portfolios.

B. SOME ANTICIPATED OBJECTIONS TO THE RESULTS

Realizing that the results of the study are likely to be criticized from many quarters, we shall now try to anticipate some of the objections and criticisms which might be expected.

Institutional frictions.—We would be the first to admit that the empirical results discussed earlier are perfectly consistent with the hypothesis that the security analysts working for mutual funds are indeed able to predict security prices somewhat; but they are prevented from realizing superior returns by institutional frictions or restrictions which prevent them from taking immediate action on their predictions. By institutional frictions we mean, for example, that analysts may not make buy-or-sell decisions themselves but must usually submit their recommendations to an "investment committee" of some sort which reviews the recommendations and makes decisions. If this process sometimes takes as long as a week or two (as has been asserted), and assuming that the analysts on the average can forecast future security prices somewhat, the empirical results imply that whenever deviations of actual price from "true" price exist, they are in general bid away very quickly—so quickly in fact that these relatively minor restrictions on buy-and-sell actions apparently remove all opportunity to earn superior returns. This should be an extremely comforting finding for the naïve investor.

The effects of large size.—There are those who claim that the sheer size of most mutual funds is such that their transactions are so large they cannot trade in most securities without significantly affecting their prices (cf. Friend *et al.* [26, pp. 361 and 387]). That is, it is asserted that in order to significantly affect the returns on a large portfolio, extremely large blocks of securities would have to be turned over in taking full account of the analysts' predictions. It is also asserted that these blocks are so large that they cannot be purchased or sold without "significantly" affecting the price of the security.

While there are no theoretical reasons why the purchase or sale of a "large" block of a particular security will not influence its price, the definition of "large" and the amount of influence on price are essentially empirical questions. Scholes [49] has examined the price effects of sales of large blocks of securities through secondary offerings and the issuance of stock rights. He finds that for 1,207 secondary offerings, the price of the securities fell on the average about 2 per cent at the time of the offering. An examination of 669 rights offerings indicates a decline in price of approximately 0.3 per cent at the time of offering. Since some of his observations represented sales of up to $185 million worth of securities, his results would certainly seem to indicate that most mutual funds could probably turn their entire portfolios over at a maximum cost of 4–5 per cent in several weeks.

Thus, if the fund managers have an ability to forecast, but are restricted from taking full advantage of their knowledge by the size of their transactions, then it must be true that it is very rare for the actual price to deviate from the "true" price by more than the transactions costs.

Legal restrictions.—Certain legal restrictions on the holdings of mutual funds may inhibit the full realization of superior forecasting ability if any exists. That is, by law the funds may not hold more than 5 per cent of their portfolios in any one security or hold more than 10 per cent of the outstanding stock of any company.

The timing of cash inflows and outflows.—It is also sometimes asserted that the funds do not do as well as might be expected because fund shareholders tend to redeem shares when market prices are low and to purchase shares when market prices are high. This argument is fallacious for two reasons: (1) Because we calculate fund returns on the basis of net asset value per share, and because all redemptions or sales of shares are executed at this net asset value (calculated at least once daily), it is impossible for the cash flows during a period to influence the returns on an outstanding share.[109] (2) Furthermore, the argument is likely to be false since it implies that one could predict the behavior of market prices on the basis of fund redemptions and sales. A vast amount of empirical work has indicated that all other attempts to create models to predict market prices have failed, and there is little reason to believe this case is different.

C. IMPLICATIONS FOR FURTHER RESEARCH

First and foremost, it is clear that the model tested here ought to be tested further on other managed portfolios, hopefully using monthly or quarterly data—pension funds, bank trusts, and university endowments would seem to be natural candidates. Moreover, the model also should be tested on unmanaged portfolios. That is, since evidence presented here indicates that the capital asset pricing model seems to have empirical as well as theoretical justification, we now need to devote a major effort to testing the capital asset pricing model on data for unmanaged portfolios and for individual securities. Work is now in progress on such a study.

[109] There is one relatively subtle way in which these cash flows might affect the returns on an outstanding share to some small degree. That is, the funds do not explicitly charge the shareholder who redeems shares (or purchases new shares) for the transactions costs involved in investing (or disinvesting) these funds. Hence current shareholders implicitly bear these costs. However, the fund obviously does not execute a transaction for every deposit and withdrawal since these are met out of a cash balance and tend to cancel each other out to a great degree.

REFERENCES

1. Aitchison, J., and Brown, J. A. C. *The Lognormal Distribution.* Cambridge: Cambridge University Press, 1957.
2. Arditti, Fred D. "Risk and the Required Return on Equity," *Journal of Finance,* XXII (March, 1967), 19–36.
3. Bank Administration Institute (Lorie, J., Cohen, K., Dean, J., Durand, D., Fisher, L., Fama, E., and Shapiro, E.). *Measuring the Investment Performance of Pension Funds.* Park Ridge, Ill.: Bank Administration Institute, 1968.
4. Blume, Marshall. "The Assessment of Portfolio Performance" (unpublished Ph.D. dissertation, University of Chicago, 1968).
5. Borch, Karl. "Indifference Curves and Uncertainty," *Swedish Journal of Economics,* LXX, No. 1 (March, 1968), 19–24.
6. Box, G. E. P., and Cox, D. R. "An Analysis of Transformations," *Journal of the Royal Statistical Society,* Series B, XXVI, No. 2 (1964), 211–43.
7. Cohen, Kalman J., and Pogue, Jerry A. "An Empirical Evaluation of Alternative Portfolio Selection Models," *Journal of Business,* XXXX (April, 1967), 166–93.
8. Cohen, Kalman J., and Pogue, Jerry A. "Some Comments Concerning Mutual Fund versus Random Portfolio Performance," *Journal of Business,* XXXXI (April, 1968), 180–90.
9. Cootner, Paul H. (ed.). *The Random Char-*

acter of Stock Market Prices. Cambridge, Mass.: M.I.T. Press, 1964.
10. Cramer, Harold. *Mathematical Methods of Statistics*. Princeton, N.J.: Princeton University Press, 1946.
11. Dietz, Peter. *Pension Funds: Measuring Investment Performance*. New York: Free Press, 1966.
12. Fama, Eugene. "The Behavior of Stock-Market Prices," *Journal of Business*, XXXVII (January, 1965), 34–105.
13. Fama, Eugene. "Portfolio Analysis in a Stable Paretian Market," *Management Science*, XI (January, 1965), 404–19.
14. Fama, Eugene, "Risk, Return, and Equilibrium: Some Clarifying Comments," *Journal of Finance* (March, 1968), 29–40.
15. Fama, Eugene. "Multi-Period Consumption-Investment Decisions," Report No. 6830. Chicago: University of Chicago, Center for Mathematical Studies in Business and Economics, June, 1968.
16. Fama, Eugene. "Risk, Return, and Equilibrium," Report No. 6831. Chicago: University of Chicago, Center for Mathematical Studies in Business and Economics, June, 1968.
17. Fama, Eugene, and Babiak, Harvey. "Dividend Policy: An Empirical Analysis," Report No. 6732. Chicago: University of Chicago, Center for Mathematical Studies in Business and Economics, November, 1967. Forthcoming in the *Journal of the American Statistical Association*.
18. Fama, Eugene, and Blume, Marshall. "Filter Rules and Stock Market Trading," *Journal of Business*, XXXIX (January, 1966), 226–41.
19. Fama, Eugene, Fisher, Lawrence, Jensen, Michael C., and Roll, Richard. "The Adjustment of Stock Prices to New Information," *International Economic Review*, X (February, 1969), pp. 1–26.
20. Fama, Eugene, and Miller, Merton. "The Theory of Valuation" (unpublished manuscript, University of Chicago, 1967).
21. Fama, Eugene, and Roll, Richard. "Some Properties of Symmetric Stable Distributions," *Journal of the American Statistical Association* (September, 1968), 817–36.
22. Farrar, Donald E. *The Investment Decision under Uncertainty*. Englewood Cliffs, N.J.: Prentice Hall, Inc., 1962.
23. Feller, William. *An Introduction to Probability Theory and Its Applications*. Vol. II. New York: John Wiley & Sons, 1966.
24. Friedman, Milton. *A Theory of the Consumption Function*. Princeton, N.J.: Princeton University Press, 1957.
25. Friedman, Milton, and Savage, L. J. "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy*, LVI (August, 1948), 279–304.
26. Friend, Irwin, Brown, F. E., Herman, Edward S., Vickers, Douglas. *A Study of Mutual Funds*. Washington, D.C.: U.S. Government Printing Office, 1962.
27. Friend, Irwin, and Vickers, Douglas. "Portfolio Selection and Investment Performance," *Journal of Finance*, XX (September, 1965), 391–415.
28. Friend, Irwin, and Vickers, Douglas. "Reevaluation of Alternative Portfolio Selection Models," *Journal of Business*, XXXXI (April, 1968), 174–79.
29. Hakansson, Nils H. "Optimal Investment and Consumption Strategies for a Class of Utility Functions," Working Paper No. 101. Los Angeles, Calif.: Western Management Science Institute, 1966.
30. Horowitz, Ira. "A Model for Mutual Fund Evaluation," *Industrial Management Review*, VI (Spring, 1965), 81–92.
31. Horowitz, Ira. "The 'Reward-to-Variability' Ratio and Mutual Fund Performanance," *Journal of Business*, XXXIX (October, 1966), 485–88.
32. Jensen, Michael C. "The Performance of Mutual Funds in the Period 1945–1964," *Journal of Finance*, XXIII (May, 1968), 389–416.
33. Johnston, J. *Econometric Methods*. New York: McGraw-Hill Book Co., 1963.
34. King, Benjamin F. "Market and Industry Factors in Stock Price Behavior," *Journal of Business*, XXXIX, Part II (January, 1966), 139–90.
35. Latané, Henry A. "Portfolio Balance—the Demand for Money, Bonds, and Stocks," *Southern Economic Journal*, XXIX (October, 1962), 71–76.
36. Lintner, John. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, XLVII (February, 1965), 13–37.
39. Lintner, John. "Security Prices, Risk, and Maximal Gains from Diversification," *Journal of Finance*, XX (December, 1965), 587–616.
38. Mandelbrot, Benoit. "The Variation of Certain Speculative Prices," *Journal of*

Business, XXXVI (October, 1963), 394–419.

39. Mandelbrot, Benoit. "Forecasts of Future Prices, Unbiased Markets and 'Martingale' Models," *Journal of Business*, XXXIX, Part 2 (January, 1966), 242–55.
40. Markowitz, Harry M. "Portfolio Selection," *Journal of Finance*, VII (March, 1952), 77–91.
41. Markowitz, Harry M. "The Utility of Wealth," *Journal of Political Economy*, LX (April, 1952), 151–58.
42. Markowitz, Harry M. *Portfolio Selection: Efficient Diversification of Investments.* ("Cowles Foundation Monograph No. 16.") New York: John Wiley & Sons, 1959.
43. Mossin, Jan. "Equilibrium in a Capital Asset Market," *Econometrica*, XXXIV (October, 1966), 768–83.
44. Mossin, Jan. "Optimal Multiperiod Portfolio Policies," *Journal of Business*, XXXXI (April, 1968), 215–29.
45. Pratt, John W. "Risk Aversion in the Small and in the Large," *Econometrica*, XXXII (January–April, 1964), 122–36.
46. Roll, Richard. "The Efficient Market Model Applied to U.S. Treasury Bill Rates" (unpublished Ph.D. dissertation, University of Chicago, 1968).
47. Samuelson, Paul A. "Proof That Properly Anticipated Prices Fluctuate Randomly," *Industrial Management Review*, VI (Spring, 1965), 41–49.
48. Samuelson, Paul A. "General Proof That Diversification Pays," *Journal of Financial Financial and Quantitative Analysis*, II (March, 1967), 1–13.
49. Scholes, Myron. "A Test of the Competitive Market Hypothesis: An Examination of the Market for New Issues and Secondary Offerings" (unpublished Ph.D. thesis proposal, University of Chicago, 1967).
50. Securities and Exchange Commission. *Public Policy Implications of Investment Company Growth.* Washington, D.C.: U.S. Government Printing Office, 1966.
51. Sharpe, William F. "A Simplified Model for Portfolio Analysis," *Management Science* (January, 1963), 277–93.
52. Sharpe, William F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, XIX (September, 1964), 425–42.
53. Sharpe, William F. "Reply," *Journal of Finance*, XX (March, 1965), 94–95.
54. Sharpe, William F. "Risk Aversion in the Stock Market," *Journal of Finance*, XX (September, 1965), 416–22.
55. Sharpe, William F. "Mutual Fund Performance," *Journal of Business*, XXXIX, Part 2 (January, 1966), 119–38.
56. Sharpe, William F. "Security Prices, Risk, and Maximal Gains from Diversification: Reply," *Journal of Finance* (December, 1966), 743–44.
57. Sharpe, William F. "Linear Programming Algorithm for Mutual Fund Portfolio Selection," *Management Science*, XIII (March, 1967), 499–510.
58. Sharpe, William F. "Reply," *Journal of Business*, XLI (April, 1968), 235–36.
59. Shelton, John P. "The Value Line Contest: A Test of the Predictability of Stock Price Changes," *Journal of Business*, XL (July, 1967), 251–69.
60. Standard & Poor's Corporation. *Trade and Securities Statistics: Security Price Index Record.* Orange, Conn.: Standard & Poor's Corporation, 1960.
61. Tobin, James. "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies*, XXV (February, 1958), 65-85.
62. Tobin, James. "The Theory of Portfolio Selection," in F. H. Hahn and F. P. R. Brechling (eds.), *The Theory of Interest Rates.* New York: St. Martin's Press, 1965.
63. Treynor, Jack L. "How To Rate Management of Investment Funds," *Harvard Business Review*, XLIII (January–February, 1965), 63–75.
64. Treynor, Jack L. "Toward a Theory of Market Value of Risky Assets" (unpublished manuscript, undated).
65. Von Neuman, John, and Morgenstern, Oscar. *Theory of Games and Economic Behavior.* Princeton, N.J.: Princeton University Press, 1953.
66. West, Richard R. "Mutual Fund Performance and the Theory of Capital Asset Pricing: Some Comments," *Journal of Business*, XLI (April, 1968), 230–34.
67. Wiesenberger, Arthur. *Investment Companies.* New York: Arthur Wiesenberger & Co., 1955 and 1965.
68. Wise, John. "Linear Estimators for Linear Regression Systems Having Infinite Variances" (unpublished paper presented at the Berkeley-Stanford Mathematical Economics Seminar, October, 1963).