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"Direct" causal cascade in the stock market

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Abstract. We use wavelets to decompose the volatility (standard deviation) of intraday (S&P500) return data across scales. We show that when investigating two-point correlation functions of the volatility logarithms across different time scales, one reveals the existence of a causal information cascade from large scales (*i.e.* small frequencies) to fine scales. We quantify and visualize the information flux across scales. We provide a possible interpretation of our findings in terms of market dynamics.

PACS. 02.50.-r Probability theory, stochastic processes, and statistics – 05.40.+j Fluctuation phenomena, random processes, and Brownian motion – 89.90.+n Other areas of general interest to physicists

1 Introduction

Modelling accurately financial price variations is an essential step underlying portfolio allocation optimization, derivative pricing and hedging, fund management and trading. The observed complex price fluctuations guide and constraint our theoretical understanding of agent interactions and of the organization of the market.

A word of caution is called for with respect to the meaning of "model" in this context and for this audience. In its broadest sense, a model (usually formulated using the language of mathematics) is a mathematical representation of a condition, process, concept, *etc.*, in which the variables are defined to represent inputs, outputs, and intrinsic states and equations or inequalities are used to describe interactions of the variables and constraints on the problem. In theoretical physics, models take a narrower meaning, such as in the Ising, Potts, ..., percolation models. In economy and finance, the term model is usually used in the broadest sense. Most of the works we refer to, including the cascade of correlations across scales found in this work, belong to this first class of models. We shall not discuss the second class of "micro-economic" models that have been developed by economists and more recently by some physicists [1], to construct artificial stock markets. In this respect, the situation is similar to a branch of the study of hydrodynamical turbulence in which "models" in the first sense aim at representing as faithfully and parsimoniously as possible the observed anomalous scaling and intermittency.

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The Gaussian paradigm of independent normally distributed price increments [2,3] has long been known to be incorrect with many attempts to improve it. Mandelbrot first proposed to use Lévy distributions [4], which are characterized by a fat tail decaying as a power law with index between 0 and 2. His suggestion arrived at an epoch when Markovitz famous mean-variance portfolio and Black-Scholes option pricing theories were being developed and widely applied. For main stream economists, the econometric nonlinear autoregressive models with conditional heteroskedasticity (ARCH) [5] and their generalizations [6] are more natural because they keep the volatility (standard deviation of price variations) as the main descriptor. Recall that heteroskedasticity refers to the fact that the variance (or volatility) is itself a stochastic variable [7]. These models address volatility clustering and partly the observed "fat tails" of distributions. The problem however is that these ARCH models capture only imperfectly the volatility correlations and the fat tails of the probability density function (pdf) of price variations. Moreover, as far as changes in time scales are concerned, the so-called "aggregation" properties of these models are not easy to control.

Recently, physicists have characterized more precisely the distribution of market price variations [8,9] and found that a power law truncated by an exponential provides a reasonable fit at short time scales (less than one day), while at larger time scales the distributions may cross over progressively to the Gaussian distribution which becomes approximately correct for monthly and larger scale price variations. Alternatively, Ghashghaie *et al.* [10] proposed a "multiplicative" cascade model based on an analogy between price dynamics and hydrodynamic turbulence. They fitted the distributions of the price variations to a superposition of Gaussians, with a log-normal distribution of the Gaussian widths.

2 Multiplicative cascade in the stock market

The controversial [9,11] analogy developed by Ghashghaie *et al.* [10] implicitly assumes that price fluctuations can be described by a *multiplicative cascade* along which the return r at a given time scale $a < T$, is given by:

$$r_a(t) \equiv \ln P(t+a) - \ln P(t) = \sigma_a(t) u(t), \qquad (1)$$

where $u(t)$ is some scale independent random variable, T is some coarse "integral" time scale and $\sigma_a(t)$ is a positive quantity that can be multiplicatively decomposed, for any decreasing sequence of scales $\{a_i\}_{i=0,\,..,n}$ with $a_0 = T$ and $a_n = a$, as [12,13]

$$\sigma_a = \prod_{i=0}^{n-1} W_{a_{i+1},a_i} \sigma_{\mathrm{T}}. \qquad (2)$$

Equation (1) together with (2) shows that the logarithm of the price is a multiplicative process. But, this is different from the simple multiplicative processes studied in [14] due to the tree-like structure of the correlations that are added by the hierarchical construction of the multiplicands.

In turbulence the field σ is related to the energy while in finance σ is called the volatility. Recall that the volatility has fundamental importance in finance since it provides a measure of the amplitude of price fluctuations, hence of the market risk. Using $\omega_a(t) \equiv \ln \sigma_a(t)$ as a natural variable, if one supposes that W_{a_{i+1},a_i} depends only on the scale ratio a_{i+1}/a_i, one can easily show, by choosing the a_i as a geometric series Ts^i ($s < 1$), that equation (2) implies that the pdf of ω at scale $a = Ts^n$ can be written as [12,13]

$$p_a(\omega) = (G_s^{\otimes n} \otimes p_{\mathrm{T}})(\omega)\ , \qquad (3)$$

where $\otimes$ means the convolution product, G_s is the pdf of $\ln W_{sa,a}$ and p_{T} is the pdf of ω_{T}. The symbol $G_s^{\otimes n} \otimes p_{\mathrm{T}}$ means that G_s has been convoluted with itself n times before being convoluted with p_{T}. This equation (3) is the exact reformulation (in log variables) of the paradigm that Ghashghaie *et al.* [10] used to fit foreign exchange (FX) rate data at different scales. Recall that it simply means that the distributions of the logarithm of the *absolute value* of the price variations can be represented by a superposition of elementary laws G_s. In this formalism, G can be proven to be the pdf of an infinitely divisible random variable [13] (hence σ is called "log-infinitely divisible"). In reference [10], G is assumed to be Normal (the cascade is called "log-normal") of variance $-\lambda^2 \ln s$.

First, let us comment on the criticisms raised by Mantegna and Stanley [11]. Note that equation (3) does not determine the shape of the pdf of the returns $r_a(t)$ *at a given scale* but specifies how this pdf changes *across scales*. For a fixed scale, the precise form for the pdf depends on both p_{T} and on the law of the variable $u(t)$ (which determines notably the sign of $r_a(t)$). Therefore, nothing prevents the pdf of $r_a(t)$ to having fat tails at small scales as observed in financial time series [10]. A cascade model actually accounts for the distribution of the volatility of returns across scales and not for the precise fluctuations of $r_a(t)$. The behavior, for $\tau > a$, of the autocorrelation function $\overline{\tilde{r}_a(t)\tilde{r}_a(t+\tau)}$ (where $\tilde{r}_a$ stands for the corresponding centered variable $\tilde{r}_a(t) = r_a(t) - \langle r_a(t)\rangle$) indeed depends on both the cascade variables and $u(t)$. For example, if $u(t)$ is a white noise, there will be no correlation between the returns while their absolute values (or the associated volatilies) are strongly correlated (see below). This is why the shape of the power spectrum of financial time series cannot be invoked as an argument against a cascade model. Moreover, as far as scaling properties of price fluctuations are concerned, it is easy to deduce from equation (3) that, if $H \ln s$ is the mean of G_s and $-\lambda^2 \ln s$ its variance, then the maximum of the pdf of $\sigma_a(t)$ varies as $a^{H-\lambda^2/2}$ (H plays the same role as the Lévy index in TLF models [8,9] with $H = 1/\mu$), while its standard deviation behaves as $a^{(H-\lambda^2)/2}$; these features are observed in both turbulence ($H \simeq 0.33$ and $\lambda^2 \simeq 0.03$) [12] and finance ($H \simeq 0.6$ and $\lambda^2 \simeq 0.015$) [10]. Therefore, as advocated in reference [10], equation (3) accounts reasonably well for one-point statistical properties of financial times series. However, because of the relatively small statistics available in finance, it is very difficult to demonstrate that equation (3) is more pertinent to fit the data than a "truncated Lévy" distribution [8,9,11].

At this point, let us emphasize that equation (2) imposes much more constraints on the statistics than equation (3) that only refers to one point statistics. The main difference between the *multiplicative* cascade model and the truncated Lévy *additive* model is that the former predicts strong correlations in the volatility while the latter assumes no correlation. It is then tempting to compute the correlations of the log-volatility ω_a at different time scales a. For that purpose, we use a natural tool to perform time-scale analysis, the *wavelet transform* (WT). Wavelet analysis has been introduced as a way to decompose signals in both time and scales [15]. The WT of $f(t) = \ln P(t)$ is defined as:

$$T_\psi[f](t,a) \equiv \frac{1}{a} \int_{-\infty}^{+\infty} f(y) \psi\left(\frac{y-t}{a}\right) \mathrm{d}y, \qquad (4)$$

where t is the time parameter, a (>0) the scale parameter and ψ the analyzing wavelet. Note that for $\psi(t) = \delta(t-1) - \delta(t)$, $T_\psi[f](t,a)$ is nothing but the return $r_a(t)$. However, in general, ψ is chosen to be well localized in both time and frequency, so that the scale a can be interpreted as an inverse frequency. Moreover, if ψ has at least two vanishing moments and χ is a bump function with $||\chi||_1 = 1$, then, the *local* volatility at scale a and time t can be defined as $\sigma_a^2(t) \equiv a^{-2} \int \chi((b-t)/a) |T_\psi(b,a)|^2 \mathrm{d}b$ [16]. Actually, thanks to the time-scale properties of the wavelet decomposition [15], when summing $\sigma_a^2(t)$ over

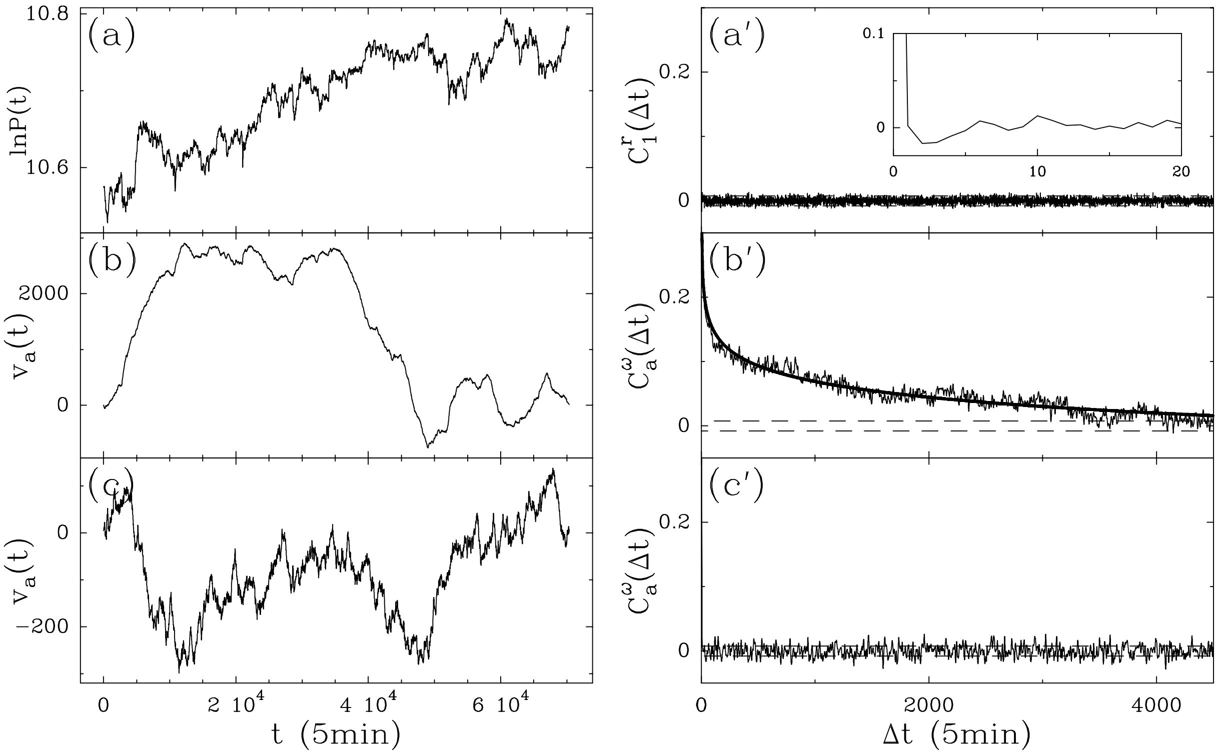

Fig. 1. a) Time evolution of $\ln P(t)$, where $P(t)$ is the S&P500 index, sampled with a time resolution $\delta t = 5$ min in the period October 1991-February 1995. The data have been preprocessed in order to remove "parasitic" daily oscillatory effects [29]. b) The corresponding "centered log-volatility walk", $v_a(t) = \sum_{i=0}^{t} \tilde{\omega}_a(i)$, as computed with the derivative of the Haar function as analyzing wavelet [15] for $a = 4$ ($\simeq 20$ min). c) $v_a(t)$ computed after having randomly shuffled the increments of the signal in (a). (a') The 5 min ($a = 1$) return correlation coefficient $C_1^r(\Delta t)$ *versus* Δt. (b') The correlation coefficient $C_a^\omega(\Delta t)$ of the log-volatility of the S&P500 at scale $a = 4$ ($\simeq 20$ min); the solid line corresponds to a fit of the data using equation (5) with $\lambda^2 \simeq 0.015$ and $T \simeq 3$ months. (c') same as in (b') but for the randomly shuffled S&P500 signal. In (a'-c') the dashed lines delimit the 95% confidence interval.

time and scale, one recovers the total square derivative of f: $\Sigma = \int\int \sigma_a^2(t)\mathrm{d}t\mathrm{d}a = \int |\mathrm{d}f/\mathrm{d}t|^2\mathrm{d}t$.

In Figure 1 are shown 3 time series for which we study the increment time correlations. Figure 1a represents the logarithm of the S&P500 index. The corresponding "centered log-volatility walk", $v_a(t) = \sum_{i=0}^{t} \tilde{\omega}_a(i)$ is represented in Figure 1b, where the symbol $\tilde{\omega}$ refers to the "centered log-volatility". Figure 1c is the same as Figure 1b but after having randomly shuffled the increments $\ln P(i+1) - \ln P(i)$ of the signal in Figure 1a. The Figure 1b with the genuine data clearly demonstrates the existence of important long-range positive temporal correlations in the log-volatility of S&P500 returns. Moreover, the statistics of $\omega_a(t)$ are found to be nearly Gaussian. However, the log-volatility walk for the "shuffled S&P500" looks very much like a Brownian motion with uncorrelated increments. This observation is sufficient to discard any additive (like TLF) model which intrinsically fails to account for the strong correlations observed in $\omega_a(t)$. The correlation coefficient $C_1^r(\Delta t) = \overline{\tilde{r}_1(t)\tilde{r}_1(t+\Delta t)}/\mathrm{var}(r_1)$ shown in Figure 1a', confirms the well-known fact that there are no correlations between the returns (except at a very small time lag as illustrated in the inset). The difference is striking in Figure 1b' where the correlation coefficient of the log-volatility walk $C_a^\omega(\Delta t) = \overline{\tilde{\omega}_a(t)\tilde{\omega}_a(t+\Delta t)}/\mathrm{var}(\omega_a)$ remains as large as 5% up to time lags corresponding to about two months. In contrast, the correlation coefficient associated to the shuffled time series in Figure 1c' is within the noise level. In sum, yesterday's or last week losses do not tell us whether we will win or lose tomorrow; but if last week, the prices changed a lot, they will on average change more than usual also tomorrow.

From the modelling of fully developed turbulent flows and fragmentation processes, random multiplicative cascade models are well known to generate long-range correlations [17–19]. We now explore whether this concept could be useful for understanding the observed long-range correlations of the volatility (and not of the price increments, which makes turbulence and financial markets drastically different in this respect). To fix ideas, let us consider a specific realization of a process satisfying equation (2). Consider the largest time scale T of the problem. We then assume that the volatility at time scale T influences the volatility of the two subperiods of length $\frac{T}{2}$ by random factors equal respectively to W_0 and W_1. In turn, each volatility over $\frac{T}{2}$ influences the two subperiods of length $\frac{T}{4}$ by random factors W_{00} and W_{10} for the first sub-period and W_{01} and W_{11} for the second one. The cascade process is assumed to continue along the time scales until

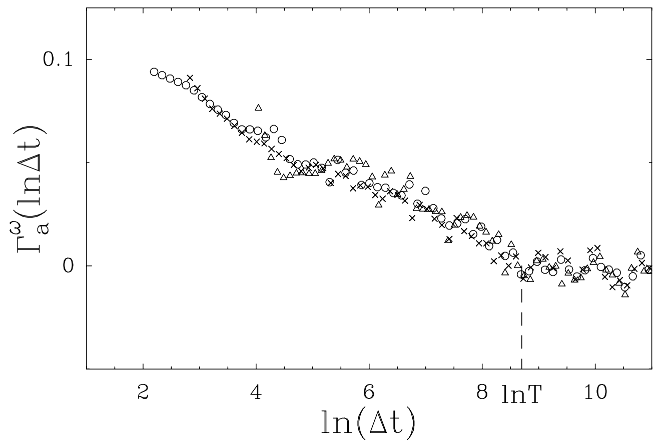

Fig. 2. The correlation function $\Gamma_a^\omega(\Delta t)$ of the log-volatility of the S&P500 index is plotted *versus* $\ln \Delta t$ for various scales a corresponding to 30 (○), 120 (×) and 480 (△) minutes. All the data collapse on a same curve which is almost linear up to an integral time scale $T \simeq 3$ months ($\ln T = 8.6$). According to equation (5), from the slope of this straight line, one gets an estimate of the parameter $\lambda^2 \simeq 0.015$.

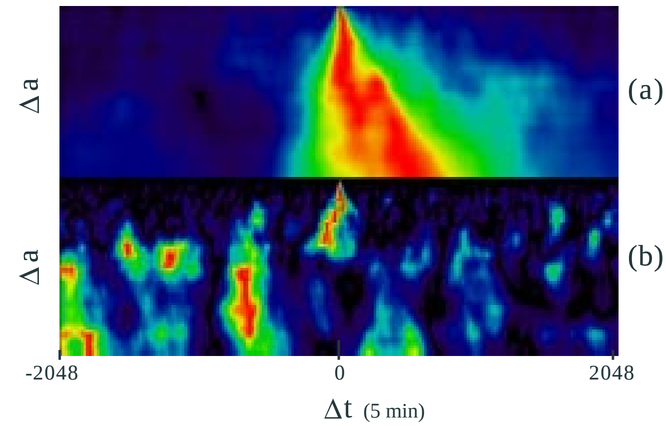

Fig. 3. The mutual information $I_a(\Delta t, \Delta a)$ (Eq. (6)) of the variables $\omega_a(t+\Delta t)$ and $\omega_{a+\Delta a}(t)$ is represented in the $(\Delta t, \Delta a)$ half-plane (5 min units); the time lag Δt spans the interval $[-2048, 2048]$ while the scale lag Δa ranges from $\Delta a = 0$ (top) to 1024 (bottom). The small scale $a = 4$ (20 minutes) is fixed. Following the horizontal axis at the top of Figure 3a, the self-information $I_a(\Delta t, 0)$ of the small scale volatility with itself is shown. At the base of Figure 3a, the mutual information between the largest scale and the small scale at a time Δt later is shown. The amplitude of $I_a(\Delta t, \Delta a)$ is coded from black for zero values to red for maximum positive values ("heat" code), independently at each scale lag Δa. (a) S&P500 index; (b) its randomly shuffled increment version. Note that, for middle scale lag values, the maxima (red spots) of the mutual information in (a) are 2 orders of magnitude larger than the corresponding maxima in (b).

the shortest tick time scale (see Ref. [13] for rigourous definitions and properties). The simplest assumption is that the factors W are i.i.d. variables with log-normal distribution of mean $-H \ln 2$ and variance $\lambda^2 \ln 2$. It is then easy to show that the correlation function averaged over a period of length T, $\Gamma_a^\omega(\Delta t) = T^{-1} \int_0^T \langle \tilde{\omega}_a(t) \tilde{\omega}_a(t+\Delta t) \rangle \, \mathrm{d}t$, can be written as [20]:

$$\Gamma_a^\omega(\Delta t) = \lambda^2 \left(\log_2 \frac{T}{\Delta t} - 2 + 2\frac{\Delta t}{T} \right) + \lambda_{\mathrm{T}}^2 \,, \qquad (5)$$

for $a \le \Delta t \le T$ ($\langle . \rangle$ means mathematical expectation and λ_{T}^2 is the variance of ω_{T}). Here, our goal is to show that the basic ingredients of this simple cascade model are sufficient to rationalize most of the features observed on the log-volatility correlations at different scales (note that one could improve this description by taking into account mutual influences of volatilities at a given scale and the possible "inverse cascade" influence of fine scales on larger ones). For $\lambda^2 \simeq 0.015$ that can be obtained independently from the fit of the pdf's, equation (5) provides a very good fit of the data (Fig. 1b') for the slow decay of the correlation coefficient with only one adjustable parameter $T \simeq 3$ months. Let us note that $C_a^\omega(\Delta t)$ can be equally well fitted by a power law $\Delta t^{-\alpha}$ with $\alpha \approx 0.2$. In view of the small value of α, this is undistinguishable from a logarithmic decay. Moreover, equation (5) predicts that the correlation function $\Gamma_a^\omega(\Delta t)$ should not depend of the scale a provided $\Delta t > a$. In Figure 2, $\Gamma_a^\omega(\Delta t)$ is plotted *versus* $\ln(\Delta t)$ for various scales a corresponding to 30, 120 and 480 min. As expected, all the data collapse on a single curve which is nearly linear up to some integral time of the order of 3 months.

Let us point out that volatility at large time intervals that cascades to smaller scales cannot do so instantaneously. From causality properties of financial signals, the "infrared" towards "ultraviolet" cascade must manifest itself in a time asymmetry of the cross-correlation coefficients $C_{a_1,a_2}^\omega(\Delta t) \equiv \mathrm{var}(\omega_{a_1})^{-1} \mathrm{var}(\omega_{a_2})^{-1} \overline{\tilde{\omega}_{a_1}(t) \times \tilde{\omega}_{a_2}(t+\Delta t)}$; in particular, one expects that $C_{a_1,a_2}^\omega(\Delta t) > C_{a_1,a_2}^\omega(-\Delta t)$ if $a_1 > a_2$ and $\Delta t > 0$. From the near-Gaussian properties of $\omega_a(t)$, the mean mutual information [21] of the variables $\omega_a(t+\Delta t)$ and $\omega_{a+\Delta a}(t)$ reads:

$$I_a(\Delta t, \Delta a) = -0.5 \log_2 \left(1 - (C_{a,a+\Delta a}^\omega(\Delta t))^2\right) . \qquad (6)$$

Since the process is causal, this quantity can be interpreted as the information contained in $\omega_{a+\Delta a}(t)$ that propagates to $\omega_a(t+\Delta t)$. In Figure 3, we have computed $I_a(\Delta t, \Delta a)$ for the S&P500 index (top) and its randomly shuffled version (bottom). One can see on the bottom picture that there is no well defined structure that emerges from the noisy background. Except in a small domain at small scales around $\Delta t = 0$, the mutual information is in the noise level as expected for uncorrelated variables. In contrast, two features are clearly visible on the top representation. First, the mutual information at different scales is mostly important for equal times. This is not so surprising since there are strong localized structures in the signal that are "coherent" over a wide range of scales. The extraordinary new fact is the appearance of a non symmetric propagation cone of information showing that the volatility a large scales influences causally (in the future) the volatility at shorter scales. Although one can also detect some information that propagates from past fine to future coarse scales, it is clear that this phenomenon is weaker than past coarse/future fine flux (the fact that the former one exists anyway suggests that a more realistic cascading process should include the causal influence of short time scales on larger ones). Figure 3 is thus a clear demonstration of the pertinence of the notion of a cascade in market dynamics. Similar features have been found on FX rates.

3 Discussion

There are several mechanisms that can be invoked to rationalize our observations, such as the heterogeneity of traders and their different time horizon [22] leading to an "information" cascade from large time scales to short time scales, the lag between stock market fluctuations and long-run movements in dividends [23], the effect of the regular release (monthly, quarterly) of major economic indicators which cascades to fine time scale. Correlations of the volatility have been known for a while and have been partially modelled by mixtures of distributions [24], ARCH/GARCH models [5] and their extensions [6]. However, as pointed out in the introduction, because they are constructed to fit the fluctuations at a given time interval, these models are not adapted to account for the above described multi-scale properties of financial time series. We have performed the same correlation analysis for simulated GARCH(1,1) processes and obtained structureless pictures similar to the one corresponding to the shuffled S&P500 in Figure 3b. Recently, Muller *et al.* [22] have proposed the HARCH model in which the variance at time t is a function of the realized variances at different scales. By construction, this model captures the lagged correlation of the volatility from the large to the small time scales. However, it does not contain the notion of cascade and involves only a few time scales. Moreover, it suffers from the same deficiencies as ARCH-type models concerning the difficulties to control and interpret parameters at different scales. Let us also mention three recent working papers by Mandelbrot *et al.* [4,25] that introduce and test a multifractal model of asset prices. Their key idea is that trading time is the cumulative (in order to be increasing) of a multifractal cascade model. In this sense, there is a strong similarity with our approach. However, in their empirical tests with real data, they do not analyze the price variations (which are the correct approximately stationary quantities) but the price itself. In our opinion, this leads to a severe loss of information and to potential distorsions, since the statistical tests become strongly perturbed by the artifical correlations produced by the cumulative process. Indeed, if $x(t)$ is a Brownian motion, $x(t)$ and $x(t+\tau)$ are strongly correlated for all τ's since they have a common history for all innovations prior to time t; this shows that the correct quantity to analyze are the price variations, as done in the present paper. Even if they construct their multifractal model by multiplicative cascades, they have not found empirically the correlation cascade that we report. In their mathematical construction, the key element is not the cascade structure but the multifractal time while the cascade is our fundamental message.

Putting together the evidence provided by the logarithmic decay of the log-volatility correlations and the volatility cascade from the infrared to the ultraviolet, we have revisited the analogy with turbulence, albeit on the *volatility* and not on the price variations. The big surprise of our work comes from the exhibition of this information cascading process: the fact that variations of prices over a few month scale influence in the future the daily price variations is extraordinarily rich of consequences. This is not so only for the fundamental understanding of the nature of financial markets but also, and maybe more important, for practical applications. Indeed, the nature of the correlations that are implied by this cascade across scales, has profound implications on the market risk, a problem of upmost concern for all financial institutions as well as individuals. In particular, these correlations are likely to have strong consequences on derivative pricing and hedging. Another very promising prospect consists in building ARCH-type processes on orthogonal wavelets basis. This work is in current progress. The present understanding with such models will allow us to calculate improved risk prices such as options, for instance using the functional formalism of reference [26] well-adapted to deal with pdf's satisfying equation (3).

It has not escaped our attention that the cascade of volatility correlations across scales discovered here has similarities to the log-periodic structures found to precede and follow large market crashes [27]. Both signatures suggest that the stock market prices are likely to display an underlying ultrametric structure. The challenge is to determine whether this results from a hierarchical structure of organization of the market or from market dynamics or both [28].

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29. If m_i and σ_i are respectively the mean and the r.m.s. of the signal within the i-th 5-min. interval of a day, the value of the signal $x(i)$ has been replaced by $m + \sigma(x(i) - m_i)/\sigma_i$.