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Scaling, self-similarity and multifractality in FX markets

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Abstract

This paper presents an empirical investigation of scaling and multifractal properties of US Dollar–Deutschemark (USD–DEM) returns. The data set is ten years of 5-min returns. The cumulative return distributions of positive and negative tails at different time intervals are linear in the double logarithmic space. This presents strong evidence that the USD–DEM returns exhibit power-law scaling in the tails. To test the multifractal properties of USD–DEM returns, the mean moment of the absolute returns as a function of time intervals is plotted for different powers of absolute returns. These moments show different slopes for these powers of absolute returns. The nonlinearity of the scaling exponent indicates that the returns are multifractal.

JEL classification: G0; G1; C1

Keywords: Scaling; Self-similarity; Multifractality; High-frequency data; Foreign exchange markets

1. Introduction

Researchers have been investigating scaling laws in finance for a long time. The beginnings may be traced back to the late 1920s.[1] At that time, the work emphasized

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[1] In the late 1920s, Elliot wave principle claimed that stock markets do not behave in a chaotic manner, but move in repetitive cycles.

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the appearance of patterns at different time scales. In the 1960s, a class of stable distributions was put forward to account for the power-law tail behavior of financial series. [2] Fractals and chaos coming from physical science led to a new wave of interest in scaling in the 1980s. [3] In recent years, the study of scaling laws resurged due to the availability of high-frequency data.

Scaling expresses invariance with respect to translation in time and change in the unit of time. That is, except for amplitude and rate of change, the rules of higher- and lower-frequency variation are the same as the rules of mid-speed frequency variation, Mandelbrot [4]. Scaling is a rule that relates returns over different sampling intervals. The shape of the distribution of returns should be the same when the time scale is changed, Calvet and Fisher [5]. In empirical studies, the scaling analysis typically exploits some kind of linear relationship between logs of variables.

In the literature, many empirical studies have shown that financial time series exhibit scaling like characteristics. Müller et al. [6] and Guillaume et al. [7] reported an empirical scaling law for mean absolute price changes over a time interval for foreign exchange rate. Dacorogna et al. [8] presented empirical scaling laws for US Dollar–Japanese Yen (USD–JPY) and British Pound–US Dollar (GBP–USD). Mantegna and Stanley [9] also found scaling behavior in the Standard and Poor index (S&P 500) by examining high frequency data. Recently, Gençay et al. [10] suggested that financial time series may not follow a single-scaling law across all horizons. They used a wavelet multi-scaling approach to show that foreign exchange rate volatilities follow different scaling laws at different horizons. They provided evidence that there was no unique global scaling in financial time series but rather scaling was time varying.

However, some literature continued to question the evidence of the scaling laws in foreign exchange (FX) markets. LeBaron [11] examined the theoretical foundation of scaling laws. He demonstrated that many graphical scaling results could have been generated by a simple stochastic volatility model. He suggested that dependence in the financial time series might be the key cause in the apparent scaling observed. LeBaron [12] presented a simple stochastic volatility model, which was able to produce visual power-laws and long memory similar to those from actual return series using comparable sample sizes. However, Stanley et al. [13] pointed out that a three-factor model cannot generate power-law behavior.

Whether or not the financial time series follow power-law and the type of scaling rule they obey are still open questions. In this paper, we will investigate intra-day US Dollar–Deutschemark (USD–DEM) returns and provide evidence that the tails of returns do follow power-law. Furthermore, the returns exhibit multifractal behavior. Section 2 is on the discussion of two types of scaling behaviors of USD–DEM returns. Namely, the behavior of the tails of the distribution of returns keeping the time interval of returns constant and the behavior of the moments of the absolute value of returns as a function of time interval. We conclude afterwards.

[2] See Mandelbrot [1] and Fama [2] for details.

[3] See Peters [3] for a survey of fractals and chaos.

2. Scaling and multifractality

2.1. Data set

The data set studied in this paper is the USD–DEM 5-min return series from January 4, 1987 21:05 Greenwich Mean Time (GMT) to December 31, 1998 21:00 GMT. The weekends from Friday 21:05 GMT to Sunday 21:00 GMT are eliminated. Therefore, there are 5 business days in a week. Foreign exchange market is a worldwide market with no business hour limitations. [4] Therefore, each day has 24 h of data. The total number of observations is 90,1152.

High frequency (less than 10-min) foreign exchange returns are reported to have three important properties: negative first-order autocorrelation, discreteness of quoted spreads, and short-term triangular arbitrage, Dacorogna et al. [8]. The autocorrelation function for USD–DEM 5-min returns up to 5 h is plotted in Fig. 1. Negative autocorrelation is observed in our data up to 20 min.

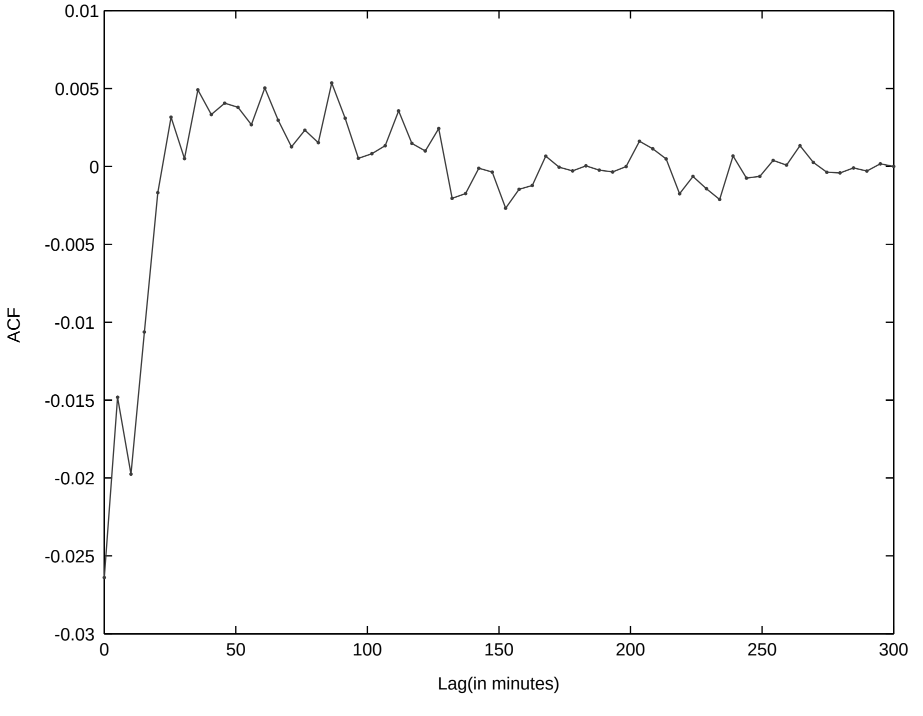

Fig. 1. 5 h lag of autocorrelation for the USD–DEM 5-min returns. Negative autocorrelation is observed up to a time lag of 20 min.

[4] Please see Dacorogna et al. [8] for a detailed examination of high-frequency financial time series.

Scaling and fractal properties of USD–DEM returns are investigated by studying the aggregate 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and daily returns. The aggregated returns are defined by

$$[r_t]_\tau = \sum_{i=1}^{\tau} r_{\tau(t-1)+i}, \quad t = 1, \ldots, 901,152/\tau , \tag{1}$$

where r_t is the original 5-min returns, $[r_t]_\tau$ represents the returns at different aggregated levels. For example, the 20-min returns are constructed by summing four 5-min returns, which $\tau = 4$. 20-min aggregate returns are defined via

$$[r_t]_4 = \sum_{i=1}^{4} r_{4(t-1)+i}, \quad t = 1, \ldots, 225,288 .$$

In the similar way, 1-h, 2-h, 4-h, 6-h, 8-h and daily returns are obtained by summing 12, 24, 48, 72, 96 and 288 5-min returns, $\tau = 12$, 24, 48, 72, 96 and 288, respectively.

Distributions of the USD–DEM 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h, 1-day returns and the normal distribution are plotted in Fig. 2 (Top). Fig. 2 (Bottom) compares distributions of the USD–DEM 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and 1-day normalized returns with the standard normal distribution. The returns are normalized by the standard deviation of the returns. The return distributions at high frequencies up to 1 day are clearly not Gaussian with pronounced peaks, thin waists and fat tails. Positive tails of USD–DEM normalized returns are shown in Fig. 3. Returns have fatter tails than the normal distribution.

2.2. Scaling

To investigate the scaling properties of the tails, cumulative distributions of the positive and negative tails for normalized 5-min returns are plotted in a log–log space (Fig. 3, Top). The straight line in the graph suggests that 5-min returns exhibit a power-law scaling in the tails. Both the negative and positive tails follow the same power-law.

When cumulative distributions of the positive and negative tails for 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and 1-day returns are plotted in log–log space in one graph (Fig. 4, Bottom), it indicates a strong similarity of the tail probabilities over several different time horizons. Both tails at each interval scale as straight lines, except for the daily ones because of the small amount of data. The straight lines are parallel to each other, which suggests that the tails at different intervals follow the same power-law.

The straight lines in these graphs present strong evidence that the USD–DEM returns exhibit a power-law scaling in the tails. This leads to an asymptotic power-law for the cumulative distribution for both the positive and negative tails in the form

$$f(r_t) \sim C r_t^{-\alpha} , \tag{2}$$

where $\sim$ means “is distributed as”, C is a prefactor and α is scaling exponent, or tail index. α can be estimated from the slopes of these straight lines. Tail indices and

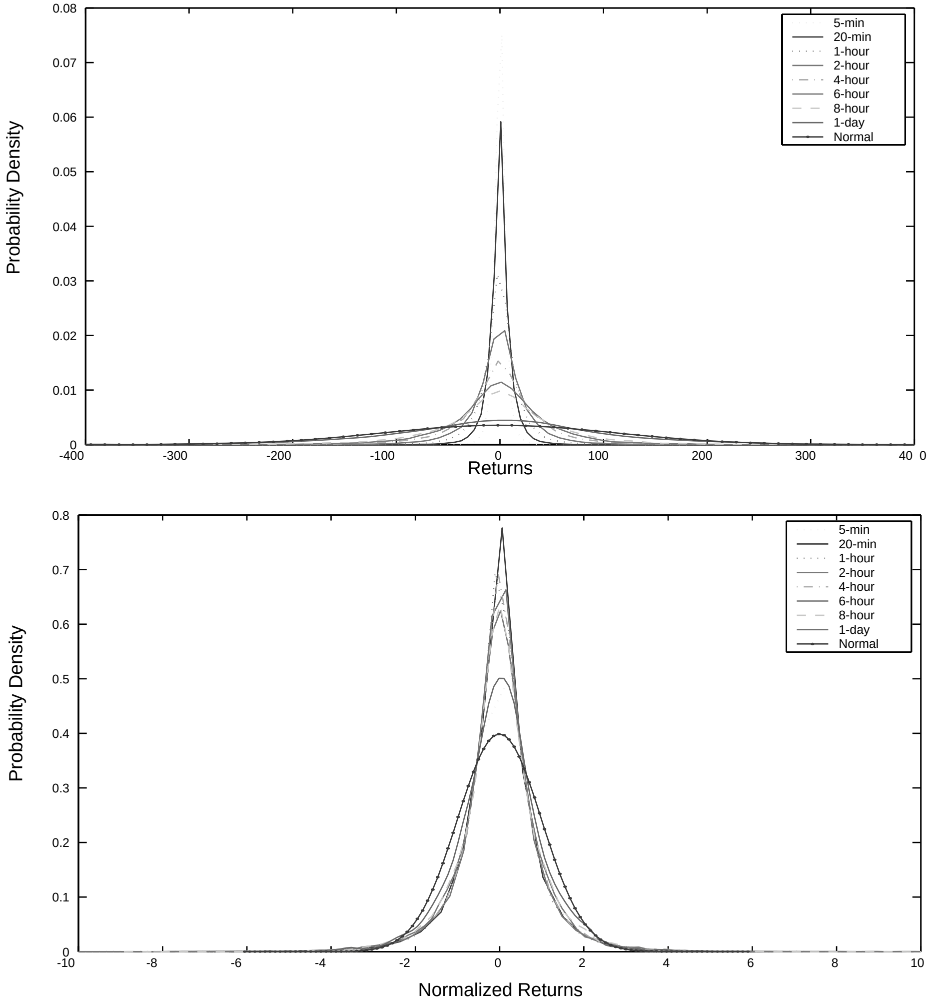

Fig. 2. Top: Probability density function of the 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h, 1-day returns and the normal distribution. Bottom: Probability density function of the 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h, 1-day normalized returns and standard normal distribution. The return distributions at high frequencies up to one day are clearly not Gaussian with pronounced peaks, thin waists and fat tails.

the corresponding bootstrap confidence intervals are presented in Table 1. The value for the tail index under aggregation is quite stable except for the daily interval where the number of data points becomes scarce for reliable estimates. Tail indices and the corresponding confidence intervals for the positive tails are plotted in Fig. 5.

The tail estimates for USD–DEM returns at different intervals are in the range 2.8777–4.4307, which contradicts the additive Lèvy walk model. The Lèvy distributions for financial fluctuations proposed in Mandelbrot [1] are subject to the limitation $0 \leqslant \alpha \leqslant 2$.

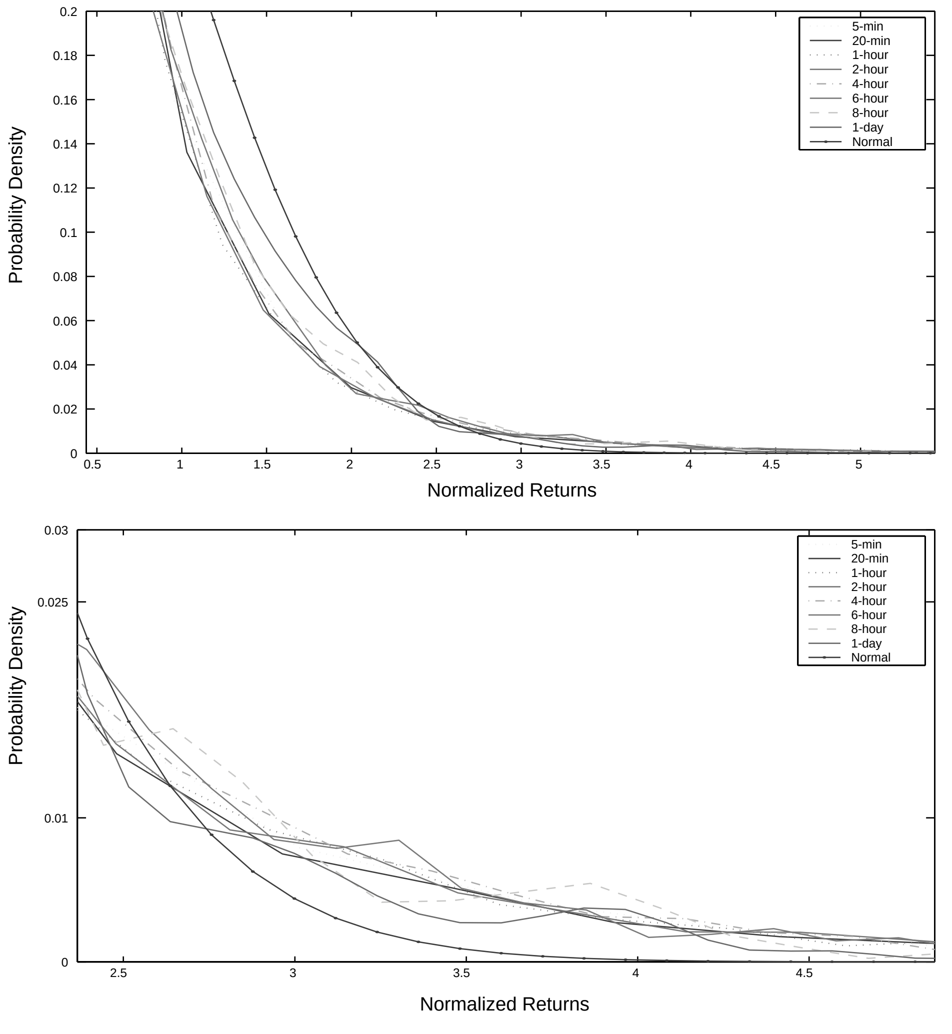

Fig. 3. Top: The positive tails of 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h, 1-day returns and standard normal distribution. Returns have fatter tails than normal distribution. Bottom: The positive tails of 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h, 1-day returns and standard normal distribution. Returns have fatter tails than normal distribution (zoomed).

The scaling exponents of USD–DEM returns are outside of the Lèvy stable region. The tails of the distribution decay faster than a Lèvy process. This contradiction was also noted by Müller et al. [14], who termed them hyperbolic non-stable distributions. The α value they obtained for several foreign exchange series range from about 3–5. They showed that the second moment was converging, whereas the fourth moment was not. The tail index for Swiss Franc (CHF), US Dollar (USD) and British Pound (GBP) estimated by Schmitt et al. [15] were in the range 3–3.6, which were also larger

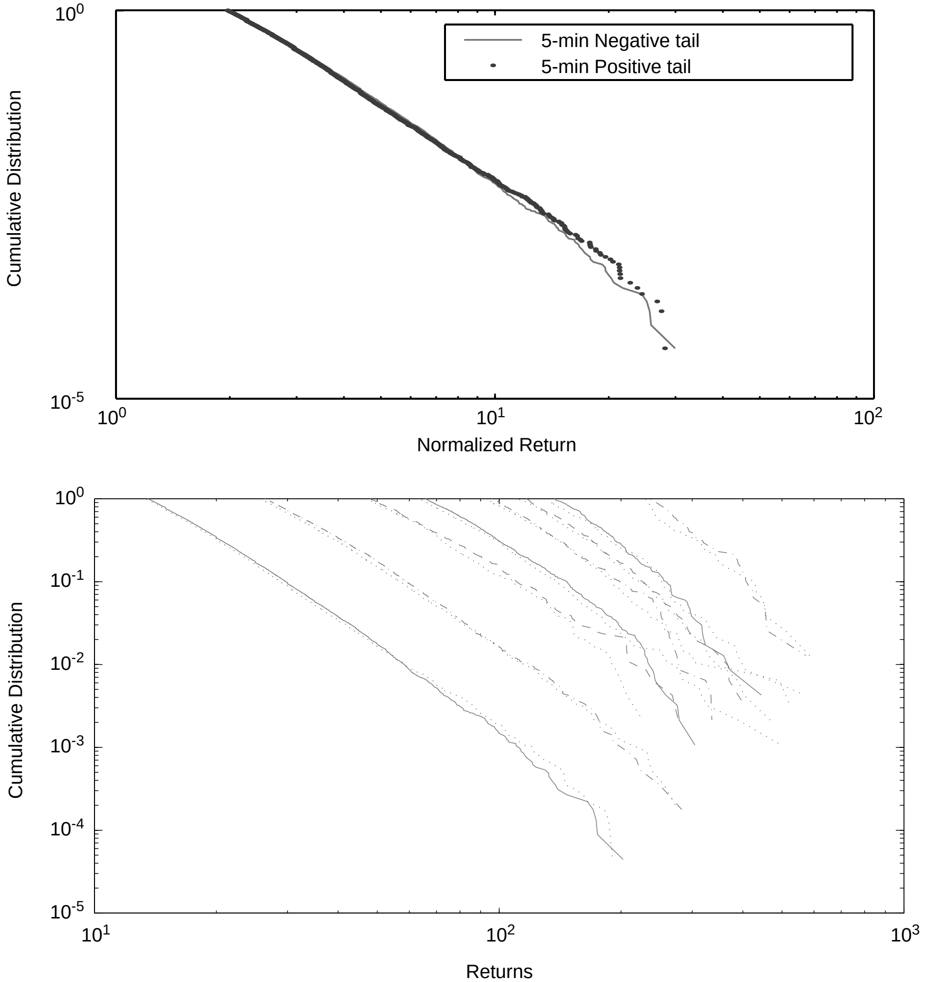

Fig. 4. Top: Cumulative distribution of the positive and negative tails for normalized 5-min returns plotted in log–log space. The returns are normalized by dividing by the standard deviation of the returns. Both tails follow power-law with similar scaling exponents. Bottom: Cumulative distribution of the positive and negative tails for 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and 1-day returns (left-to-right) plotted in log–log space. Both tails at each interval scale as straight lines, except for the daily ones because of the small amount of data. The straight lines parallel to each other, which suggests that the tails at different intervals follow the same power-law.

than 2. Liu et al. [16] showed a power-law behavior with exponent $\alpha \approx 3$ also held for both the S&P 500 index and individual companies.

In general, the tails of all possible distributions can be classified into three categories:

1. Thin-tailed distributions for which all moments are finite and whose cumulative distribution function declines exponentially in the tails,

Table 1
The tail indices and the corresponding bootstrap confidence intervals of 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and 1-day returns

	5-min (NT)	5-min (PT)	20-min (NT)	20-min (PT)
$\hat{\alpha}$	3.0910	3.1020	3.0285	3.0328
CI	[3.0824, 3.0984]	[3.0959, 3.1092]	[3.0092, 3.0497]	[3.0161, 3.0501]
	1-h (NT)	1-h (PT)	2-h (NT)	2-h (PT)
$\hat{\alpha}$	2.8777	3.0722	3.2602	3.1640
CI	[2.7977, 2.9779]	[2.9796, 3.2045]	[3.1721, 3.3584]	[3.1250, 3.2105]
	4-h (NT)	4-h (PT)	6-h (NT)	6-h (PT)
$\hat{\alpha}$	3.5493	3.4509	3.7506	3.6236
CI	[3.4267, 3.6871]	[3.3808, 3.5249]	[3.5717, 3.9025]	[3.5420, 3.7260]
	8-h (NT)	8-h (PT)	1-day (NT)	1-day (PT)
$\hat{\alpha}$	4.1853	3.6890	4.4307	4.0175
CI	[3.9835, 4.3844]	[3.5508, 3.8362]	[4.0081, 4.8112]	[3.7372, 4.2737]

The value for the tail index under aggregation is quite stable except for the daily interval where the number of data points becomes scarce for reliable estimates. NT represents negative tail and PT corresponds to positive tail. CI is the abbreviation of confidence interval.

2. Fat-tailed distributions whose cumulative distribution function declines with a power in the tails, and
3. Thin-tailed distributions with finite tails.

These categories can be distinguished by the use of only one parameter, the tail index α with $\alpha = \infty$ for distributions of category (1), $\alpha > 0$ for category (2), and $\alpha < 0$ for category (3). The distributions with tail index $\alpha = \infty$ include the normal, exponential, gamma and log-normal distributions where only the log-normal distribution has a moderately heavy-tail. The distributions with tail index $\alpha < 0$ are the thin-tailed distributions such as uniform and beta distributions which do not have much power in explaining financial time series.

Tail index α gives the critical order of divergence of moments. $\alpha \approx 3$ indicates that USD–DEM returns belong to the class of fat-tailed non-stable distributions which have a finite tail index. The mean and variance are well-defined for the returns. The kurtosis clearly diverges and the behavior of the skewness is not very clear. The distributions of returns are not normally distributed, but have fat tails. The two tails of the distribution decay more slowly than a Gaussian. There are more observations far away from the mean than is the case in a normal distribution. Fat tails imply that extraordinary losses occur more frequently than in a normal distribution.

2.3. *Multifractality*

Similarity of some empirical properties of USD–DEM returns at different scales suggests fractal behavior. Fractals, one of the most useful discoveries in mathematics,

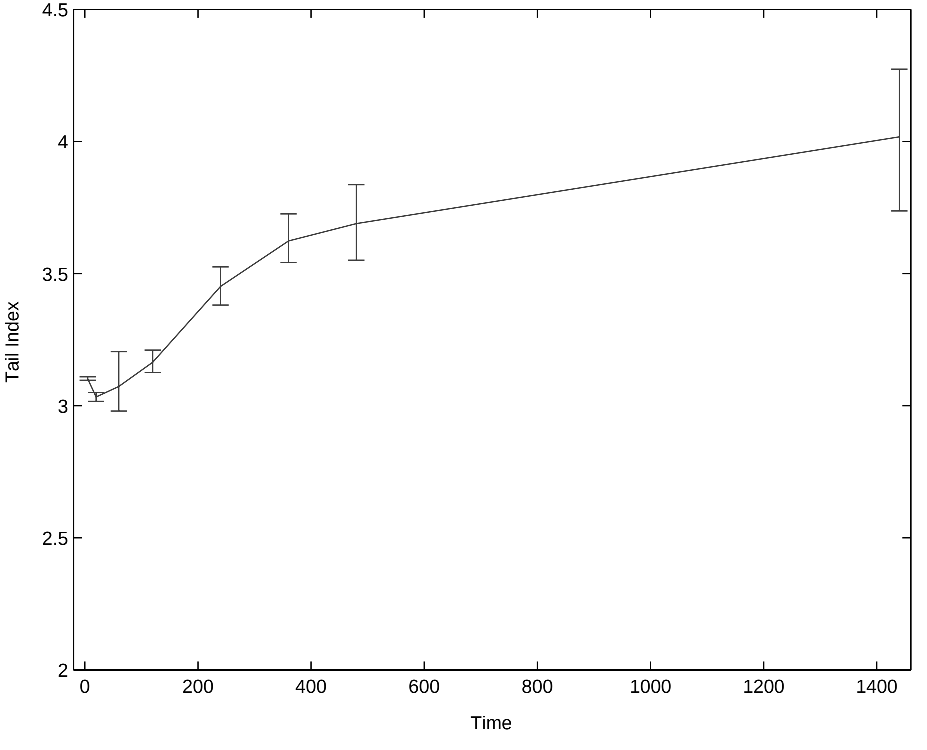

Fig. 5. The tail indices and the corresponding bootstrap confidence intervals for the positive tails of 5-min, 20-min, 1-h, 2-h, 4-h, 6-h, 8-h and 1-day returns.

were first introduced into the finance area by Mandelbrot. A fractal is an object in which the parts are in some way related to the whole. Self-similarity, an invariance with respect to scaling, is an important characteristic of fractals. It means that the object or process is similar at different scales. Each scale resembles the other scales, but is not identical. For example, individual branches of a tree are qualitatively self-similar to the other branches, but each branch is also unique. A self-similar object appears unchanged after increasing or shrinking its size. A stochastic process is said to be self-affine, or self-similar if

$$\{[r_{t_1}]_\tau, \ldots, [r_{t_k}]_\tau\} \overset{\mathrm{d}}{=} \{\tau^H [r_{t_1}], \ldots, \tau^H [r_{t_k}]\} \ . \tag{3}$$

The exponent H, called self-affinity index, or scaling exponent, of $[r_t]$, satisfies $0 < H < 1$. The operator, $\overset{\mathrm{d}}{=}$, indicates that the two probability distributions are equal. It means that samplings at different intervals yield the same distribution for the process $[r_t]$ subject to a scale factor. Strictly speaking, self-affinity and self-similarity are not identical. [5] In this paper the term self-similarity is used.

[5] See Mandelbrot et al. [17] for the discussion of the difference between self-affine and self-similarity.

A self-similar process is also called uniscaling or unifractal. Multifractal (or multi-scaling) process extends the idea of similarity to allow more general scaling functions. Multifractality is a form of generalized scaling that includes both extreme variations and long-memory [15].

As discussed in Mandelbrot et al. [17], a self-similar process satisfies the simple scaling rule

$$[r_t]_\tau \stackrel{\mathrm{d}}{=} \tau^H [r_t] \; . \tag{4}$$

If different values of H are found in different intervals, the process is more likely to be multifractal rather than self-similar. The theory of multifractals examines a more general relationship

$$[r_t]_\tau \stackrel{\mathrm{d}}{=} \tau^{H(\tau)} [r_t] \; , \tag{5}$$

where $H(\tau)$ is a function of τ. Multifractality can be understood from different aspects. Scaling properties in moments of the process are the most common way to study multifractality. In this paper, the mean moment of absolute returns as a function of time intervals for different power of absolute returns are examined in order to test the multifractal properties of USD–DEM returns

$$E(|[r_t]_\tau|^q) \sim c(q) T^{\xi(q)} \; , \tag{6}$$

where E is the expectation operator, T represents time interval, $T = 5\tau$, q is the order of moments, $c(q)$ and $\xi(q)$ are both deterministic functions of q. The function $\xi(q)$ is called the scaling function of the multifractal process. Unifractal or uniscaling is a special case of multifractal which has a linear scaling function. Multifractal processes are characterized by the non-linearity of functions $\xi(q)$.

To investigate the multifractal properties of USD–DEM returns, the mean moment of the absolute returns as a function of time intervals for several different values of q are plotted in a double logarithmic space. Fig. 6 (Top) plots $E(|[r_t]_\tau|^q)$ against time intervals for different values of q. The time intervals range from 5 min to one day. From bottom to top on the graph, the values of q increase from 0.5 to 2.5. The straight lines in the figure indicate that the scaling of Eq. (6) is very well respected. It shows a power-law scaling with time intervals, that is, the qth moment of the returns subject to the scaling factor when moving from a high-frequency interval to low-frequency interval. For example, variance of hourly returns can be rescaled to variance of daily returns by considering the scaling factor. The moments show different slopes for different values of q, which suggests different scaling laws for different order of moments. Slope increases corresponding to an increasing q. The return process is monofractal if $\xi(q)$ is a linear function of q and multifractal if $\xi(q)$ is nonlinear. In Fig. 6 (Top), the variation of the line's slope with q is nonlinear, suggesting multifractal behavior of the return process. To examine the properties of exponent function, the scaling function is plotted against q (Fig. 6, Bottom). The nonlinearity of $\xi(q)$ verifies that USD–DEM return process is multifractal.

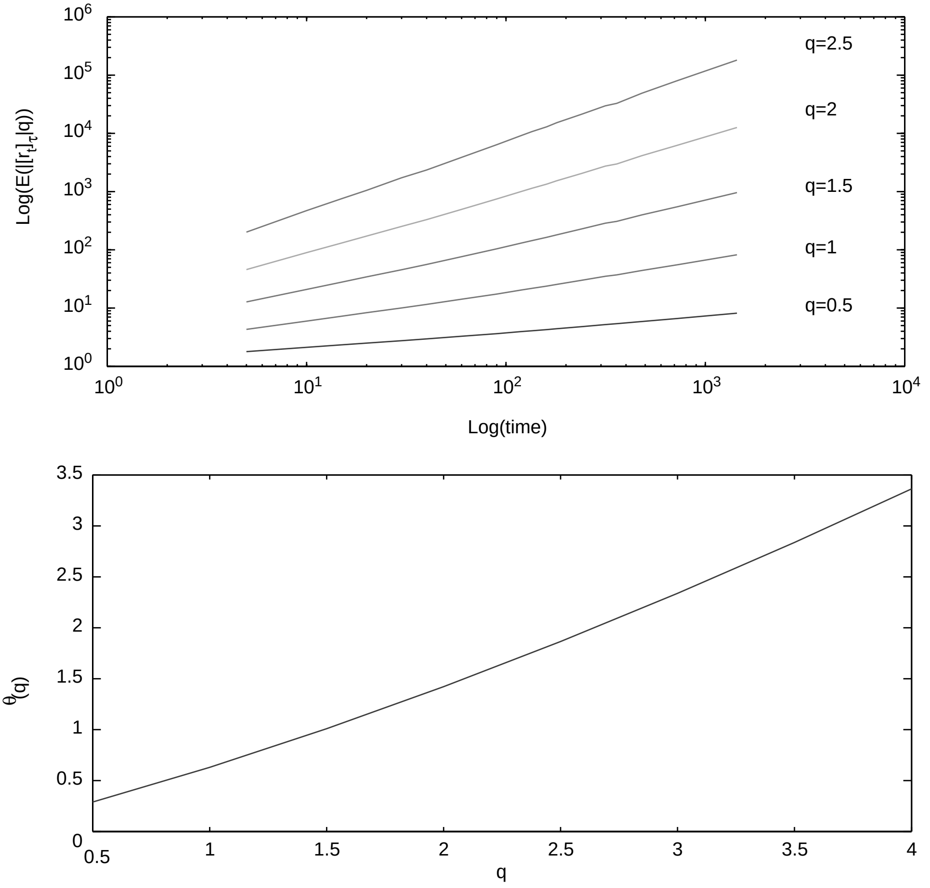

Fig. 6. Top: The mean moment of the absolute returns $E(|[r_t]_\tau|^q)$ plotted as a function of time intervals for several different values of q ranging from 0.5 (bottom) to 2.5 (top). In log–log space these appear to approximate straight lines which suggests a power-law behavior. Slopes are different for different value of q. Bottom: The estimated scaling exponents of moments plotted as a function of q. Nonlinearity of the scaling exponent function suggests multifractal behavior.

The nonlinearity of $\xi(q)$ is tested by plotting $E(|[r_t]_\tau|^q)/E(|[r_t]_\tau|)^q$ versus time intervals for several different values of q ranging from 0.5 (bottom) to 4 (top) in a log–log space. Fig. 7 shows $E(|[r_t]_\tau|^q)/E(|[r_t]_\tau|)^q$ is an approximate linear function of τ for $q < 2$. The straight lines which are observed have a slope of $\xi(q) - q\xi(1)$. In the case that $\xi(q)$ is a linear function of q, the slopes of the straight lines should be 0. However, if $\xi(q)$ is nonlinear, a trend should be expected. The graph shows the slopes of the straight lines are non-zero except for $q = 1$. When $q < 1$, the slope is positive. When $q > 1$, negative slopes are observed. Therefore, $\xi(q)$ is a nonlinear function of q, which provides further evidence that USD–DEM return process is multifractal.

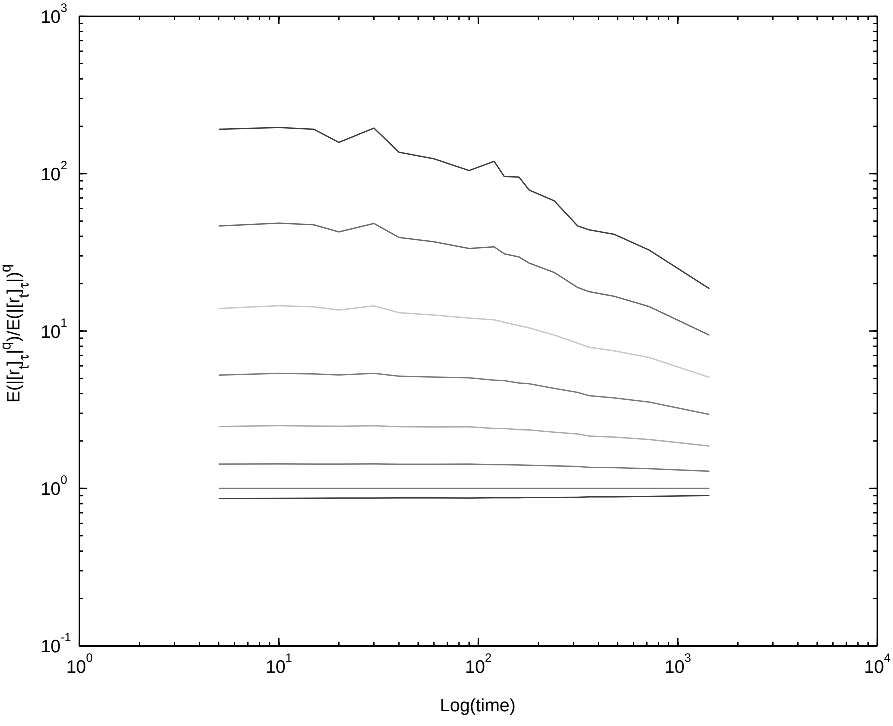

Fig. 7. A test of the nonlinearity of $\xi(q)$. The ratio $E(|[r_t]_\tau|^q)/E(|[r_t]_\tau|)^q$ plotted as a function of time intervals for several different values of q ranging from 0.5 (bottom) to 4 (top) in a log–log space. The approximate straight lines observed have slopes of $\xi(q) - q\xi(1)$, which should be zero in case of linearity. Obviously, the slopes are not zero except when $q = 1$, suggesting the nonlinearity of $\xi(q)$. This indicates USD–DEM returns are multifractal.

3. Conclusions

This paper has investigated the scaling, self-similarity and multifractal properties of USD–DEM returns. Scaling properties of USD–DEM returns are examined for the negative and positive tails of returns. Both tails are parallel shifts of each other over different time intervals, which indicates self-similarity in USD–DEM returns.

However, USD–DEM returns are not self-similar fractals. Instead, they follow a multifractal scaling law. The relationship of the mean moment of absolute returns and time intervals at different orders of moment are examined. The linear relationship between the mean moments and time intervals indicates the scaling properties of absolute returns. The nonlinearity of the scaling exponent provides evidence for multifractal properties of USD–DEM returns.

Acknowledgements

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References

[1] B. Mandelbrot, The variation of certain speculative prices, J. Business 36 (1963) 394–419.
[2] E.F. Fama, The behavior of stock market prices, J. Bus. 38 (1965) 34–105.
[3] E.E. Peters, Chaos and Order in The Capital Markets: A New View of Cycles, Prices, and Market Volatility, Wiley, New York, 1991.
[4] B. Mandelbrot, Fractals and Scaling in Finance: Discontinuity, Concentration, Risk, Springer, New York, 1997b.
[5] L. Calvet, A. Fisher, Multifractality in Asset Returns: Theory and Evidence, 2001, Manuscript.
[6] U. Müller, M. Dacorogna, O.V. Pictet, M. Schwarz, C. Morgenegg, Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis, J. Banking Financ. 14 (1990) 1189–1208.
[7] D.M. Guillaume, M. Dacorogna, U. Müller, O.V. Pictet, From the bird's eye to the microscope: a survey of new stylized facts of the intra-daily foreign exchange markets, Financ. Stoch. 1 (1997) 95–129.
[8] M.M. Dacorogna, R. Gençay, U.A. Müller, R.B. Olsen, O.V. Pictet, An Introduction to High-Frequency Finance, Academic Press, San Diego, 2001.
[9] R.N. Mantegna, H.E. Stanley, Scaling behavior in the dynamics of an economic index, Nature 376 (1995) 46–49.
[10] R. Gençay, F. Selçuk, B. Whitcher, Scaling properties of foreign exchange volatility, Physica A 289 (2001) 249–266.
[11] B. LeBaron, Volatility persistent and apparent scaling laws in finance, Brandeis University, 1999, Manuscript.
[12] B. LeBaron, Stochastic volatility as a simple generator of apparent financial power laws and long memory, Quant. Financ. 1 (2001) 621–631.
[13] H.E. Stanley, V. Plerou, Scaling and universality in economics: empirical results and theoretical interpretation, Quant. Financ. 1 (2001) 563–567.
[14] U. Müller, M. Dacorogna, O.V. Pictet, Fractals and intrinsic time—A challenge to econometricians, Internal Document UAM. 1993-08-16, Olsen & Associates, Zurich, Switzerland, 1995.
[15] F. Schmitt, D. Schertzer, S. Lovejoy, Multifractal analysis of foreign exchange data, Appl. Stochastic Model Data Anal. 15 (1999) 29–53.
[16] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, K. Peng, H.E. Stanley, Statistical properties of the volatility of price fluctuations, Phys. Rev. E 60 (1999) 1390–1400.
[17] B. Mandelbrot, A. Fisher, L. Calvet, A multifractal model of asset returns, Cowles Foundation Working Paper No. 1164, 1997a.