# Scaling and universality in economics: empirical results and theoretical interpretation

*H Eugene Stanley and Vasiliki Plerou* comment on the paper by Blake LeBaron, on page 621 of this issue, by examining the degree to which the twin concepts of scaling and universality apply to economic systems as compared with other physical systems comprising a large number of interconnected and interacting components.

The purpose of this comment, inspired by [1], is to offer some perspective on the criteria used to establish the credibility of power-law behaviour, and to offer some thoughts on the interpretation of such behaviour when it exists.

That at least *some* economic phenomena are described by power laws has been recognized for over 100 years since Pareto investigated the statistical character of the wealth of individuals by modelling them using the scale-invariant distribution

$$f(x) \sim x^{-\alpha} \qquad (1)$$

where $f(x)$ denotes the number of people having income $x$ or greater than $x$, and $\alpha$ is an exponent that Pareto estimated to be 1.5 [2]. Pareto noticed that his result was universal in the sense that it applied to nations 'as different as those of England, of Ireland, of Germany, of the Italian cities, and even of Peru'. A physicist would say that the universality class of the scaling law (1) includes all the aforementioned countries as well as Italian cities, since by definition two systems belong to the same universality class if they are characterized by the same exponents.

In the century following Pareto's discovery, the twin concepts of scaling and universality have proved to be important in a number of scientific fields [3, 4]. A striking example was the elucidation of the puzzling behaviour of systems near their critical points. Over the past few decades it has come to be appreciated that the scale-free nature of fluctuations near critical points also characterizes a huge number of diverse systems also characterized by strong fluctuations. This set of systems includes examples that, at first sight, are as far removed from physics as is economics. For example, consider the percolation problem, which in its simplest form consists of placing blue pixels on a fraction $p$ of randomly chosen plaquettes of a yellow computer screen (figure 1). A remarkable fact is that the largest connected component of blue pixels magically spans the screen at a threshold value $p_c$. This purely geometrical problem has nothing to do with the small branch of physics called critical-point phenomena. Nonetheless, the fluctuations that occur near $p = p_c$ are scale free and functions describing various aspects of the incipient spanning cluster that appears at $p = p_c$ are described by power laws. Indeed, the concepts of scaling and universality provide the conceptual framework for understanding this geometry problem.

> Recent attempts to make models that reproduce the empirical scaling relationships suggest that significant progress on understanding firm growth may be well underway

It is becoming clear that almost any system comprised of a large number of interacting units has the potential of displaying power-law behaviour. Since economic systems are in fact comprised of a large number of interacting units having the potential of displaying power-law behaviour, it is perhaps not unreasonable to examine economic phenomena within the conceptual framework of scaling and universality [3–8]. We will discuss this topic in detail below.

So having embarked on a path guided by these two theoretical concepts, what does one do? Initially, critical phenomena research—guided by the Pareto principles of scaling and universality—was focused on finding which systems display scaling phenomena, and on discovering the actual values of the relevant exponents. This initial empirical phase of critical phenomena research proved vital, for only by carefully obtaining empirical values of exponents such as $\alpha$ could scientists learn which systems have the same exponents (and hence belong to the same *universality class*). The fashion in which physical systems partition into disjoint universality classes proved essential to later theoretical developments such as the renormalization group [6]—which offered some insight into the reasons why scaling and universality seem to hold; ultimately it led to a better understanding of the critical point.

Similarly, the initial research in economics guided by the Pareto principles has largely been concerned with establishing which systems display scaling phenomena, and with measuring the numerical values of the exponents with sufficient accuracy that one can begin to identify universality classes if they exist. Economics

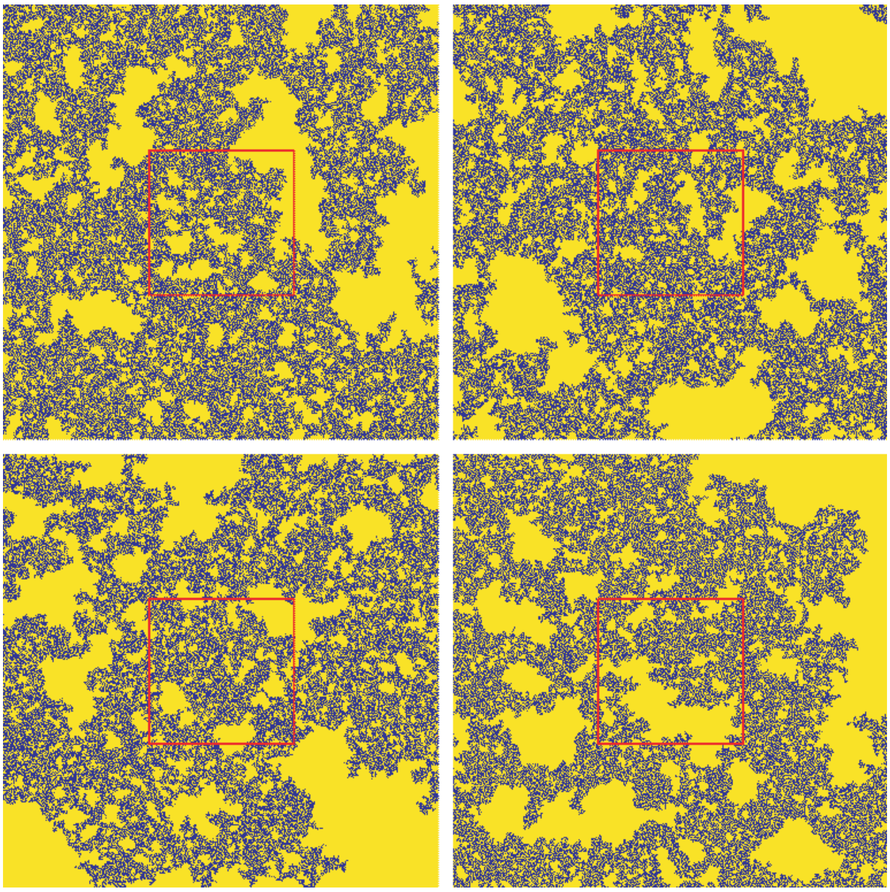

**Figure 1.** We can experience the striking self-similarity of a fractal when we examine a series of pictures of a large percolation cluster created at the percolation threshold $p = p_c$. A little box is cut out of the first picture, blown up, and used as the second picture. The same little box procedure can be repeated in the second picture, creating the third picture, and in the third, creating the fourth. The untrained eye immediately recognizes that the statistical properties in all four pictures are the same, and to confirm this by a simple experiment, we can remove the labels, mix the pictures up, and then see how long it takes to put them back into sequence. It takes a remarkably long time and, significantly, can be carried out only by searching for *non-statistical* features of the patterns, such as specific invaginations of a specific part of the cluster. An educational game is to time how long it takes each player to detect by eye which of the 24 possible panel orderings is the correct one that arranges the four panels in increasing order of magnification. This figure is courtesy of J Kantelhardt.

systems differ from often-studied physical systems in that the number of subunits are considerably smaller in contrast to macroscopic samples in physical systems that contain a huge number of interacting subunits, as many as Avogadro's number, $6 \times 10^{23}$. In contrast, in an economic system, one initial work was limited to analysing time series comprising of order of magnitude $10^3$ terms, and nowadays with high-frequency data the standard, one may have $10^8$ terms. Scaling laws of the form of (1) are found that hold over a range of a factor of $\approx 10^6$ on the $x$-axis [9–11]. Moreover, these scaling laws appear to be universal in that they, like the Pareto scaling law, hold for different countries [12] and for other social organizations [13, 14].

**The question of how to interpret these different power laws remains open**

Recent attempts to make models that reproduce the empirical scaling relationships suggest that significant progress on understanding firm growth may be well underway [15–18], leading to the hope of ultimately developing a clear and coherent 'theory of the firm'. One use of the recent empirical work is that now any acceptable theory must respect the fact that power laws hold over typically six orders of magnitude; as Axtell put the matter rather graphically: 'the power-law distribution is an unambiguous target that any empirically accurate theory of the firm must hit' [9].

With this background on power laws and scale invariance in geometry and in economics, we turn now to [1], which concerns the well-studied problem of finance fluctuations, where a consistent set of empirical facts is beginning to emerge. One fact that has been confirmed by numerous, mostly independent, studies is that stock price fluctuations are characterized by a scale-invariant cumulative distribution function of the power-law form (1) with $\alpha \approx 3$ [19, 20]. This result is also universal, in the sense that this inverse cubic law exponent is within the error bars of results for different segments of the economy, different time periods, and different countries—and is the same for stock averages as different as the S&P and the Hang Seng [21]. Many commodity prices appear to have exponent $\alpha \approx 3$ also [22], with the exception of cotton fluctuations [3] for which $\alpha \approx 1.7$; this difference is of some interest because only if $\alpha < 2$ is the distribution of the Lévy form.

Other quantities characterizing stock movements (such as the volatility, share volume traded and number of trades) also display a range of power-law behaviour over a range of typically $10^2$. The exponents characterizing the power-law decays are different for different quantities [23–26]; it is tempting to conjecture that in finance there may exist a set of relations among the power-law exponents found, just as there exist relations among the exponents characterizing different quantities near the critical point. Finally, it is well known that while the autocorrelation function of price returns decays rapidly, the autocorrelation function of the absolute values of price returns is power-law correlated in time (see [23] and extensive earlier work cited therein).

The question of how to interpret these different power laws remains open. The three-factor stochastic volatility process presented in [1] possesses three built-in scales at values that one can choose at will, so does not generate power-law behaviour, but instead generates a strongly curved plot on log–log paper which is never even approximately linear, even if the range is as small as that shown by the straight line in figures 2–5 of [1] (a factor of $\approx 3$—fully 30 times smaller than the domain of power-law behaviour displayed by empirical data [20, 23]).

To test the possibility that facts might support a conclusion which is the opposite of what the title of [1] claims, we compare in figure 2 the model with empirical data. We deliberately choose the same number of points to analyse for both. Two different sample sizes are shown, 500 000 (the size chosen in figures 2–5 of [1] and hence also used to analyse empirical data) and 12 000 000 (the size of the empirical data set, and hence also used for the model). One thereby obtains a sense of the dependence of the results on the size of the data set. We use the TAQ database listing all transactions of all stocks over the 2-year period 1994–1995) [20,23]. Both plots are consistent with equation (1), with $\alpha \approx 3$ ('the inverse cubic law'); a regression fit from 2 to 80 standard deviations yields $\alpha = 3.10 \pm 0.03$. Plots for the model are not at all linear, but rather display a strong downward curvature, strikingly seen since all plots lie well above every possible chord—in contrast to the empirical data which do not consistently lie above chords. Accordingly, there is no evidence—in contrast to the claim of [1]—that the model displays any region of approximate power law behaviour; additionally, there is no evidence that the model resembles the empirical data.

Newcomers to the field of scale invariance often ask why a power law does not extend 'forever' as it would for a mathematical power law of the form $f(x) = x^{-\alpha}$. This legitimate concern is put to rest by by reflecting on the fact that power laws for natural phenomena are not equalities, but rather are asymptotic relations of the form $f(x) \sim x^{-\alpha}$. Here the tilde denotes *asymptotic equality*. Thus $f(x)$ is not 'approximately equal to' a power law so the notation $f(x) \approx x^{-\alpha}$ is

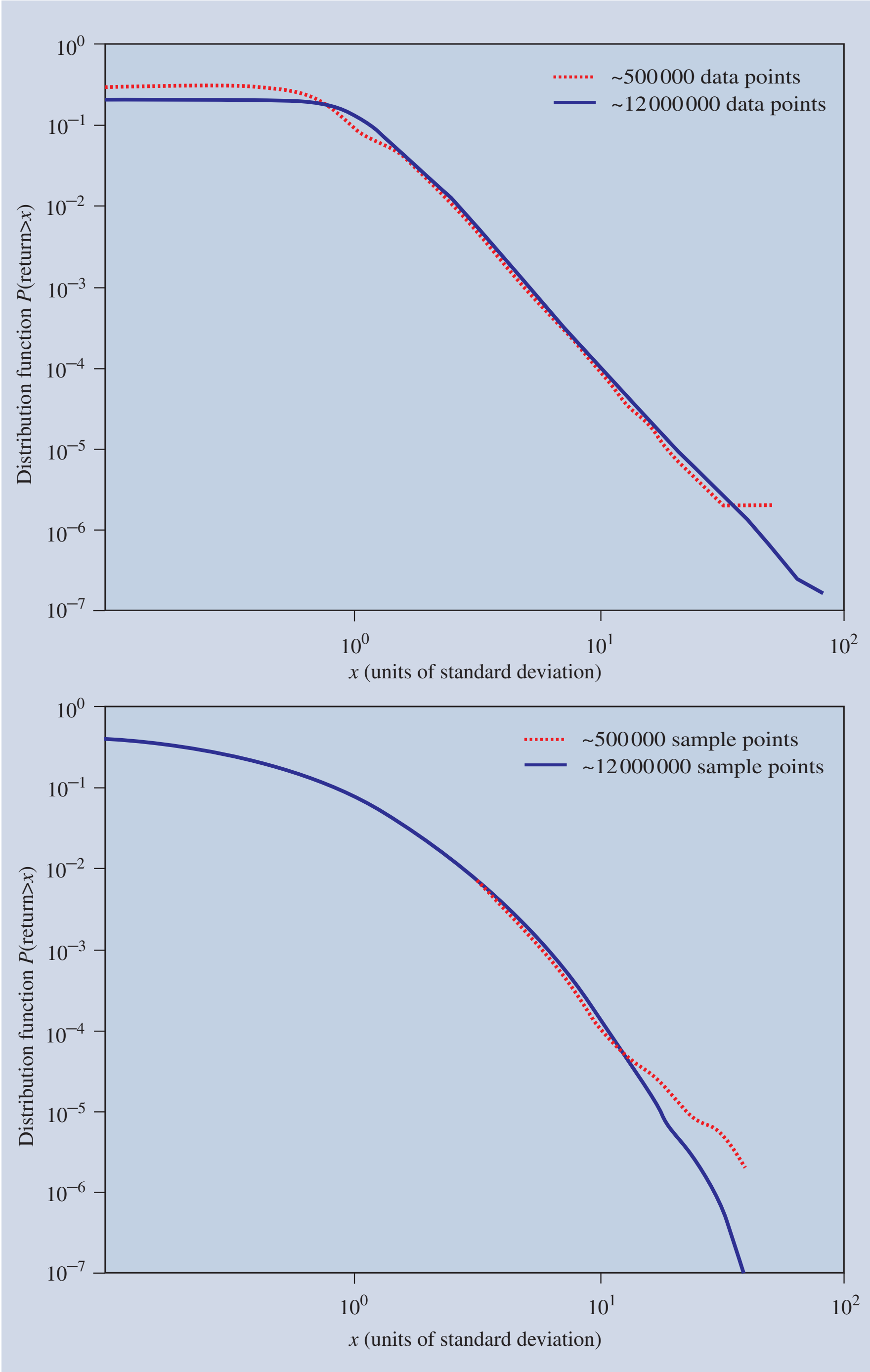


**Figure 2.** The empirical cumulative distribution function of stock price fluctuations (the probability that the stock price change is larger than $x$, where $x$ is measured in units of standard deviations). This figure is designed to compare the model proposed in [1],with empirical data, first for the identical number, 500 000, of statistical samples used in [1] and then for 24 times as many samples, 12 000 000. The bottom panel displays no hint of linearity, even over a small region of the plot. The top panel shows that even the smaller 500 000 data sample displays a region of clear linearity ($\approx$ one order of magnitude) and this power-law regime *increases* for the larger 12 000 000 sample and becomes almost two orders of magnitude (see also figure 2 of [20], which analyses $\approx$ 40 000 000 records at 5 minute intervals). That the region of linearity *decreases* on successive improvements of the model, yet *increases* on successive improvements of the data set and demonstrates conclusively that the 3-factor stochastic volatility model fails a reasonable test for power-law behaviour, while in contrast the empirical data pass this test.

inappropriate. Similarly, $f(x)$ is not proportional to a power law, so the notation $f(x) \propto x^{-\alpha}$ is also inappropriate. Rather, asymptotic equality means that $f(x)$ becomes increasingly like a power law as $x \to \infty$. Moreover, crossovers abound in financial data, such as the characteristic crossover from power-law behaviour to simple Gaussian behaviour as the time horizon $\Delta t$ over which fluctuations are calculated increases; such crossovers are also characteristic of other scale-free phenomena in the physical sciences [4], where the Yule distribution often proves quite useful.

For reasons of this sort, standard statistical fits to data are inappropriate, and often give distinctly erroneous values of the exponent $\alpha$. Rather, one reliable way of estimating the exponent $\alpha$ is to form successive slopes of pairs of points on a log–log plot, since these successive slopes will be monotonic and converge to the true asymptotic exponent $\alpha$. In the case of figure 2, one finds that successive slopes for the empirical data converge rapidly to a value $\alpha \approx 3$ while successive slopes for the model diverge. In summary, we cannot support the claim in the title of [1] that the model is a 'simple generator of financial power laws and long memory' when in light of the above results it appears that the model does not generate power laws and does not agree with empirical data which do follow a power law.

Our experience in discussing [1] with colleagues has often led to the first reaction that the model agrees with the empirical data shown, at least over the limited range of a factor of three on the abscissa. While it is clear that a three-factor model cannot generate power-law behaviour, it is less clear why the empirical data analysed appear at first glance to be well approximated by the model. The first fact is that the region of linearity of the data is not so large as in typical modern studies because the total quantity of data analysed is not that large, since only a low-frequency time series comprising daily data is used. Only 28 094 records are analysed (not $4 \times 10^7$ as in recent studies [20, 23]) and the model simulations are presented for limited sample size. The second fact is that when one superposes a curved line (the model) on a straight line (the data), the untrained eye is easily tempted to find agreement where none exists—and closer inspection of figures 2–5 of [1] actually reveals a rather poor agreement between model and data due to the pronounced downward curvature of the model.

> It is tempting to imagine that there might be analogies between finance and known physical processes displaying similar scale-invariant fluctuations

Before concluding, we ask what sort of understanding could eventually develop if one takes seriously the power laws that appear to characterize finance fluctuations. It is tempting to imagine that there might be analogies between finance and known physical processes displaying similar scale-invariant fluctuations. For example, if one measures the wind velocity in turbulent air, one finds intermittent fluctuations that display some similarities with finance fluctuations [27]. However these similarities are not borne out by quantitative analysis—e.g., one finds non-Gaussian statistics and intermittency for both turbulence fluctuations and stock price fluctuations, but the time evolution of the second moment and the shape of the probability density functions are different for turbulence and for stock market dynamics [28, 29].

More recent work pursues a rather different analogy: phase transitions in spin systems. Stock prices respond to fluctuations in demand, just as the magnetization of an interacting spin system responds to fluctuations in the magnetic field. Periods with large number of market participants buying the stock imply mainly positive changes in price, analogous to a magnetic field causing spins in a magnet to align. Recent work [30] quantifies the relations between price change and demand fluctuations, and finds results reminiscent of phase transitions in spin systems, where the divergent behaviour of the response function at the critical point (zero magnetic field) leads to large fluctuations [4].

Since the evidence for an analogy between stock price fluctuations and magnetization fluctuations near a critical point is backed up by quantitative analysis of finance data, it is legitimate to demand a theoretical reason for this analogy. To this end, we discuss briefly one possible theoretical understanding for the origin of scaling and universality in economic systems. As mentioned above, economic systems consist of interacting units just as critical point systems consist of interacting units. Two units are correlated in what might seem a hopelessly complex fashion—consider, e.g. two spins on a lattice, which are correlated regardless of how far apart they are. The correlation between two given spins on a finite lattice can be partitioned into the set of all possible topologically linear paths connecting these two spins—indeed this is the starting point of one of the solutions of the two-dimensional Ising model (see appendix B of [4]). Since correlations decay exponentially along a one-dimensional path, the correlation between two spins would at first glance seem to decay exponentially. Now it is a mathematical fact that the total number of such paths grows exponentially with the distance between the two spins—to be very precise, the number of paths is given by a function which is a product of an exponential and a power law. The constant of the exponential *decay* depends on temperature while the constant for the exponential *growth* depends only on geometric properties of the system [4]. Hence by tuning temperature it is possible to achieve a threshold temperature where these two 'warring exponentials' balance each other, and a previously negligible power-law factor that enters into the expression for the number of paths will dominate. Thus power-law scale invariance emerges as a result of cancelling exponentials, and universality emerges from the fact that the interaction paths depend not on the interactions but rather on the connectivity. Similarly, in economics, two units are correlated through a myriad of different correlation paths; 'everything depends on everything else' is the adage expressing the intuitive fact that when one firm changes, it influences other firms. A more careful discussion of this argument is presented, not for the economy but for the critical phenomena problem, in [6].

Finally, a word of humility with respect to our esteemed economics colleagues is perhaps not inappropriate. Physicists may care passionately if there are analogies between physics systems they understand (like critical point phenomena) and economics systems they do not understand. But why should anyone else care? One reason is that scientific understanding of earthquakes moved ahead after it was recognized that extremely rare events—previously regarded as statistical outliers requiring for their interpretation a theory quite distinct from the theories that explain everyday shocks—in fact possess the identical statistical properties as everyday events; e.g., all earthquakes fall on the same straight line on an appropriate log–log plot. Since economic phenomena possess the analogous property, the challenge is to develop a coherent understanding of financial fluctuations that incorporates not only everyday fluctuations but also those extremely rare 'financial earthquakes'.

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*H Eugene Stanley and Vasiliki Plerou are at the Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*