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Pricing of index options under a minimal market model with log-normal scaling

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Abstract

This paper describes a two-factor model for a diversified market index using the growth optimal portfolio with a stochastic and possibly correlated intrinsic timescale. The index is modelled using a time transformed squared Bessel process with a log-normal scaling factor for the time transformation. A consistent pricing and hedging framework is established by using the benchmark approach. Here the numeraire is taken to be the growth optimal portfolio. Benchmarked traded prices appear as conditional expectations of future benchmarked prices under the real world probability measure. The proposed minimal market model with log-normal scaling produces the type of implied volatility term structures for European call and put options typically observed in real markets. In addition, the prices of binary options and their deviations from corresponding Black–Scholes prices are examined.

1. Introduction

A rich literature has now emerged on continuous time equity index modelling. For some recent accounts one can refer to Renault and Touzi (1996), Musiela and Rutkowski (1997), Rebonato (1999), Schönbucher (1999), Fouque *et al* (2000), Lewis (2000), Rosenberg (2000), Balland (2002) and Barndorff-Nielsen and Shephard (2002).

Despite much effort there is still no commonly accepted market index model. Over the years a substantial body of empirical evidence has been accumulated on the dynamics of indices and their derivatives. Recent empirical research is documented, for instance, in Franks and Schwartz (1991), Heynen (1993), Heynen *et al* (1994), Bakshi *et al* (1997), Dumas *et al* (1997), Das and Sundaram (1999), Skiadopolous *et al* (2000), Tompkins (2001) and Cont and da Fonseca (2002).

It is evident from principal component analysis, see Cont and da Fonseca (2002), that a one-factor model should be able to account for 75–95% of the movements of the index and implied volatility surface. However, an additional factor is needed to capture, say, about 90–99% of these dynamics. Thus, a two-factor model is likely to be required for most applications. In addition, at present a two-factor model represents the limit of what can be reliably implemented to produce fast and accurate pricing tools, see Brigo and Mercurio (2001). For these reasons we will focus our attention on a class of two-factor models and will discuss a particular choice for these factors.

We will specify the factors in a parsimonious way such that the most important features of the observed index dynamics are captured when using a few, piecewise constant parameters. By using a result in Platen (2003b), which states that a diversified portfolio approximates the *growth optimal portfolio* (GOP) we identify the dynamics of a diversified index with that of the GOP.

For the index model considered here, the GOP itself and the scaling of the intrinsic GOP time are chosen as the two primary factors. Together with the short rate they determine the dynamics of the GOP, which plays a central role in our model formulation, see Platen (2003b).

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In the literature, one typically assumes the existence of an equivalent risk neutral measure. We relax this assumption to enable us to choose from a wider range of modelling alternatives. The approach is demonstrated for a generalization of the *minimal market model* (MMM) described in Platen (2001, 2002), where the discounted GOP follows a time transformed squared Bessel process. The scaling of the time transformation is modelled by a possibly correlated log-normal process. A consistent *fair pricing* concept is obtained by using the GOP as numeraire or *benchmark*, see Long (1990) and Platen (2002). This pricing method assumes that benchmarked traded derivative price processes are martingales under the real world measure. Thus, benchmarked derivative prices equal the conditional expectations of their future benchmarked prices. They can be computed as expectations under the real world probability measure.

We study the pricing of zero coupon bonds as well as European call and put options and binary options. These calculations enable us to compute the implied volatility term structures for European call options and to compare these to those typically observed in real markets. Finally, we consider the pricing of path dependent binary options and examine the differences between these prices and corresponding Black–Scholes prices.

2. A benchmark model

2.1. Primary security accounts

Let us introduce a *financial market model* as a particular case of the continuous, complete benchmark model proposed in Platen (2002).

We introduce the *primary security account processes* $S^{(0)}$, $S^{(1)}, \ldots, S^{(d)}$, which are typically share accounts. The price of the jth primary security account at time t, when measured in units of the domestic currency, is denoted by $S^{(j)}(t)$ for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. We assume that in the jth primary security account the accrued income or loss from holding this security is always reinvested. We assume that $S^{(j)}(t)$ is the strong, unique solution of the stochastic differential equation (SDE)

$$\mathrm{d}S^{(j)}(t) = S^{(j)}(t)\left\{a^j(t)\,\mathrm{d}t + \sum_{k=1}^{d} b^{j,k}(t)\,\mathrm{d}W^k(t)\right\} \tag{2.1}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$ with $S^{(j)}(0) > 0$. In this framework the standard Wiener processes $W^k = \{W^k(t), t \in [0, T]\}$, $k \in \{1, 2, \ldots, d\}$ are defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0, \infty)$, fulfilling the usual conditions, see Protter (1990). Here the filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ models the evolution of market information over time, where $\mathcal{A}_t$ denotes the market information available at time $t \in [0, T]$.

The jth *appreciation rate* $a^j = \{a^j(t),\ t \in [0, T]\}$ and (j, k)th *volatility* $b^{j,k} = \{b^{j,k}(t),\ t \in [0, T]\}$ are considered to be $\underline{\mathcal{A}}$-adapted stochastic processes for $j \in \{0, 1, \ldots, d\}$ and $k \in \{1, 2, \ldots, d\}$, see Protter (1990). We set $a^0(t) = r(t)$ and $b^{0,k}(t) = 0$ for $k \in \{1, 2, \ldots, d\}$ so that $S^{(0)}(t)$ is the value of the savings account at time t, where $r(t)$ is the short term interest rate at time t. Furthermore, the *volatility matrix* $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ is for Lebesgue-almost-every $t \in [0, T]$ assumed to be *invertible*. This ensures a proper securitization of the uncertainty generated by the Wiener processes $W^1, \ldots,$ W^d and makes the resulting model complete, see Platen (2002).

We denote by $S = \{S(t) = (S^{(0)}(t), S^{(1)}(t), \ldots, S^{(d)}(t))^\top,$ $t \in [0, T]\}$ the vector of primary security accounts. Here $A^\top$ denotes the transpose of a vector or matrix A. By introducing the appreciation rate vector $a(t) = (a^1(t), \ldots, a^d(t))^\top$ and the unit vector $\mathbf{1} = (1, \ldots, 1)^\top$, we obtain the *market price for risk* vector

$$\begin{aligned}\theta(t) &= (\theta^1(t), \ldots, \theta^d(t))^\top \\ &= b^{-1}(t)[a(t) - r(t)\mathbf{1}]\end{aligned} \tag{2.2}$$

for $t \in [0, T]$. The notion (2.2) allows us to rewrite the SDE (2.1) in the form

$$\mathrm{d}S^{(j)}(t) = S^{(j)}(t)\left\{r(t)\,\mathrm{d}t + \sum_{k=1}^{d} b^{j,k}(t)[\theta^k(t)\,\mathrm{d}t + \mathrm{d}W^k(t)]\right\} \tag{2.3}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$.

2.2. Strategies

Let us now consider portfolios of primary security accounts. We say that a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0, T]\}$ is a *strategy* if δ is S-integrable, see Protter (1990). The jth component $\delta^{(j)}(t) \in (-\infty, \infty)$ of the strategy δ denotes the number of units of the jth primary security account, which are held at time $t \in [0, T]$ in the corresponding portfolio, $j \in \{0, 1, \ldots, d\}$. For a strategy δ we denote by $S^{(\delta)}(t)$ the value of the corresponding portfolio at time t, when measured in units of the domestic currency. This means that

$$S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t) S^{(j)}(t) \tag{2.4}$$

for $t \in [0, T]$. A strategy δ and the corresponding portfolio process $S^{(\delta)}$ are called *self-financing* if

$$\mathrm{d}S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\,\mathrm{d}S^{(j)}(t) \tag{2.5}$$

for all $t \in [0, T]$. For a self-financing strategy δ no outflow or inflow of funds occur in the corresponding portfolio $S^{(\delta)}$. All changes in the value of this portfolio are due to corresponding gains from trade using the primary security accounts. We will consider in the following only self-financing strategies and corresponding self-financing portfolios. Therefore, we omit from now on the word ‘self-financing’.

For a given strategy δ the corresponding portfolio value $S^{(\delta)}(t)$ satisfies according to (2.5) and (2.3) the SDE

$$\begin{aligned}\mathrm{d}S^{(\delta)}(t) &= S^{(\delta)}(t) r(t)\,\mathrm{d}t \\ &\quad + \sum_{k=1}^{d}\sum_{j=0}^{d} \delta^{(j)}(t) S^{(j)}(t) b^{j,k}(t)(\theta^k(t)\,\mathrm{d}t + \mathrm{d}W^k(t))\end{aligned} \tag{2.6}$$

for $t \in [0, T]$.

2.3. Growth optimal portfolio

We now introduce the GOP with value $S^{(\delta_*)}(t)$ at time $t \in [0, T]$, see Kelly (1956), Long (1990), Karatzas and Shreve (1998) or Platen (2002). The GOP is the portfolio that maximizes the growth rate, that is the drift of $\log(S^{(\delta)}(t))$ for all $t \in [0, T]$. The optimal strategy $\delta_* = \{\delta_*(t) = ({\delta_*}^{(0)}(t), {\delta_*}^{(1)}(t), \dots, {\delta_*}^{(d)}(t))^\top, t \in [0, T]\}$ follows in a straightforward manner from solving the first order conditions for the quadratic growth rate maximization problem, see Karatzas and Shreve (1998). The GOP then satisfies the SDE

$$\mathrm{d}S^{(\delta_*)}(t) = S^{(\delta_*)}(t)\left[r(t)\,\mathrm{d}t + \sum_{k=1}^{d}\theta^k(t)(\theta^k(t)\,\mathrm{d}t + \mathrm{d}W^k(t))\right] \tag{2.7}$$

for $t \in [0, T]$ with

$$S^{(\delta_*)}(0) > 0. \tag{2.8}$$

It can be seen from (2.7) and (2.2) that the volatilities $\theta^k(t)$, $k \in \{1, 2, \dots, d\}$, of the GOP are the market prices for risk. Note that the GOP dynamics is fully characterized by the market prices for risk process $\theta^k = \{\theta(t)^k, t \in [0, T]\}$, $k \in \{1, 2, \dots, d\}$, and the short rate process $r = \{r(t), t \in [0, T]\}$.

2.4. Benchmarked prices

Throughout the following we use the GOP as *numeraire* or *benchmark* and call prices when expressed in units of $S^{(\delta_*)}(t)$ *benchmarked prices*. The jth *benchmarked primary security account price* at time t is then

$$\hat{S}^{(j)}(t) = \frac{S^{(j)}(t)}{S^{(\delta_*)}(t)} \tag{2.9}$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. For a portfolio $S^{(\delta)}$ we introduce its *benchmarked portfolio value*

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} \tag{2.10}$$

at time $t \in [0, T]$. By application of the Itô formula together with (2.6) and (2.7), the benchmarked portfolio value $\hat{S}^{(\delta)}(t)$ satisfies the SDE

$$\mathrm{d}\hat{S}^{(\delta)}(t) = \sum_{k=1}^{d}\sum_{j=0}^{d}\delta^{(j)}(t)\hat{S}^{(j)}(t)(b^{j,k}(t) - \theta^k(t))\,\mathrm{d}W^k(t) \tag{2.11}$$

for $t \in [0, T]$. The right-hand side of (2.11) is driftless. Therefore, the benchmarked portfolio process $\hat{S}^{(\delta)}$ is under appropriate integrability assumptions on σ, b and θ an $(\underline{\mathcal{A}}, P)$-martingale, see Karatzas and Shreve (1991), where P is the real world probability measure.

2.5. Fair pricing

We work here in a more general framework than what is provided by the standard risk neutral pricing methodology. In particular, we do not insist on the existence of an equivalent risk neutral martingale measure. This enlarges crucially the range of models that we can choose from. To price derivatives we introduce the following concept of *fair pricing*.

Definition 2.1. *A value process* $U = \{U(t), t \in [0, T]\}$, *with*

$$E\left(\frac{|U(t)|}{S^{(\delta_*)}(t)}\right) < \infty$$

for $t \in [0, T]$, *is called fair if the corresponding benchmarked value process* $\hat{U} = \{\hat{U}(t) = \frac{U(t)}{S^{(\delta_*)}(t)}, t \in [0, T]\}$ *forms an* $(\underline{\mathcal{A}}, P)$*-martingale, that is*

$$\hat{U}(t) = E(\hat{U}(\tau)|\mathcal{A}_t) \tag{2.12}$$

for $0 \leqslant t \leqslant \tau \leqslant T$.

A benchmarked fair price is the expected value of any of its future benchmarked prices. For a fair price process its last observed benchmarked value is thus the best forecast of any of its future benchmarked values.

Let us define a *contingent claim* H_τ that matures at a stopping time $\tau \in [0, T]$ as an $\mathcal{A}_\tau$-measurable payoff with

$$E\left(\frac{|H_\tau|}{S^{(\delta_*)}(\tau)}\middle|\mathcal{A}_t\right) < \infty \tag{2.13}$$

for all $t \in [0, \tau]$. The corresponding fair price process $U_{H_\tau} = \{U_{H_\tau}(t), t \in [0, \tau]\}$ for this contingent claim must satisfy at the stopping time τ the condition

$$U_{H_\tau}(\tau) = H_\tau. \tag{2.14}$$

Thus the corresponding fair derivative price process, when benchmarked, must be an $(\underline{\mathcal{A}}, P)$-martingale, see definition 2.1. Consequently, its benchmarked value $\hat{U}_{H_\tau}(t)$ is at time $t \in [0, \tau]$ given by the conditional expectation

$$\hat{U}_{H_\tau}(t) = \frac{U_{H_\tau}(t)}{S^{(\delta_*)}(t)} = E(\hat{U}_{H_\tau}(\tau)|\mathcal{A}_t). \tag{2.15}$$

Therefore, the fair contingent claim price $U_{H_\tau}(t)$ at time t, when expressed in units of the domestic currency, is given by the *fair pricing formula*

$$U_{H_\tau}(t) = S^{(\delta_*)}(t)E\left(\frac{H_\tau}{S^{(\delta_*)}(\tau)}\middle|\mathcal{A}_t\right) \tag{2.16}$$

for $t \in [0, \tau]$. It is straightforward to show that if an equivalent risk neutral martingale measure exists, then the fair pricing formula generalizes the well-known risk neutral pricing formula, see Platen (2002).

2.6. Discounted GOP

Let us discount the GOP value $S^{(\delta_*)}(t)$ at time t by the savings account value $S^{(0)}(t)$, see the remarks following (2.1). The discounted GOP

$$\bar{S}^{(\delta_*)}(t) = \frac{S^{(\delta_*)}(t)}{S^{(0)}(t)} \tag{2.17}$$

satisfies by application of the Itô formula (2.7) and (2.1) the SDE

$$\mathrm{d}\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(t)\sum_{k=1}^{d}\theta^k(t)(\theta^k(t)\,\mathrm{d}t + \mathrm{d}W^k(t)) \tag{2.18}$$

for $t \in [0, T]$. Here the *total market price for risk* $|\theta(t)|$ or GOP volatility is given by the expression

$$|\theta(t)| = \sqrt{\sum_{k=1}^{d} (\theta^k(t))^2} \tag{2.19}$$

for $t \in [0, T]$. Let us introduce a new parameter process $\alpha = \{\alpha(t), t \in [0, T]\}$, called the *discounted GOP drift*

$$\alpha(t) = \bar{S}^{(\delta_*)}(t)|\theta(t)|^2 \tag{2.20}$$

for $t \in [0, T]$. The parametrization given in (2.20) leads to the total market price for risk

$$|\theta(t)| = \sqrt{\frac{\alpha(t)}{\bar{S}^{(\delta_*)}(t)}}. \tag{2.21}$$

Thus, by (2.18), (2.20) and (2.21) we obtain for the discounted GOP the SDE

$$d\bar{S}^{(\delta_*)}(t) = \alpha(t)\, dt + \sqrt{\alpha(t)\bar{S}^{(\delta_*)}(t)}\, d\hat{W}(t) \tag{2.22}$$

with

$$d\hat{W}(t) = \frac{1}{|\theta(t)|} \sum_{k=1}^{d} \theta^k(t)\, dW^k(t), \tag{2.23}$$

for $t \in [0, T]$, see Platen (2002). Using Levy's theorem, see Karatzas and Shreve (1991), it can be shown that $\hat{W} = \{\hat{W}(t), t \in [0, T]\}$ is a standard Wiener process on $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$.

2.7. GOP time

Under the given parametrization the discounted GOP in (2.22) turns out to be a particular diffusion process. To see this, we introduce the *GOP time*

$$\varphi(t) = \tfrac{1}{4} \int_0^t \alpha(s)\, ds \tag{2.24}$$

for $t \in [0, T]$. Then the discounted GOP process $X = \{X(\varphi), \varphi \in [0, \varphi(T)]\}$ with

$$X(\varphi(t)) = \bar{S}^{(\delta_*)}(t) \tag{2.25}$$

satisfies in GOP time the SDE

$$dX(\varphi) = 4\, d\varphi + 2\sqrt{X(\varphi)}\, d\hat{W}_\varphi \tag{2.26}$$

for $\varphi \in [0, \varphi(T)]$ with $X(0) = \bar{S}^{(\delta_*)}(0) > 0$, where

$$d\hat{W}_{\varphi(t)} = \tfrac{1}{2}\sqrt{\alpha(t)}\, d\hat{W}(t) \tag{2.27}$$

for $t \in [0, T]$, see Platen (2003b). It follows from (2.26) that the discounted GOP process X is in GOP time a *squared Bessel process* of dimension *four*, see Revuz and Yor (1999). Therefore, the discounted GOP process $\bar{S}^{(\delta_*)} = \{\bar{S}^{(\delta_*)}(t), t \in [0, T]\}$ in (2.22) is a time transformed squared Bessel process of dimension four, see Platen (2002). It is important to see that the discounted GOP is, in fact, the time transform of a fundamental diffusion process with explicitly known transition density. Here it should be emphasized that we did not impose any major modelling assumptions on the primary security dynamics.

Of course, in general, the GOP time will be random. To construct a particular model one has to specify the discounted GOP drift process $\alpha = \{\alpha(t), t \in [0, T]\}$. For instance, when $|\theta(t)|^2$ in (2.20) is a deterministic function of time, then we obtain a Black–Scholes type dynamics for the GOP. However, there is general empirical evidence that the volatility is stochastic. Also, if the discounted GOP drift is assumed to be deterministic one obtains a stochastic volatility model that is still quite realistic, see Heath and Platen (2002). To model stochastic volatility with a second factor we specify the discounted GOP drift in a particular way.

3. MMM with log-normal scaling

3.1. Basic formulation

According to Platen (2003a) the GOP can be interpreted as a diversified accumulation index for the stock market. To model the discounted index with a more realistic GOP time we now consider a generalization of the MMM proposed in Platen (2001). The MMM incorporates the structure previously described by using a squared Bessel process of general dimension $\nu > 2$, see (2.22) or (2.26).

For this version of the MMM the discounted GOP process $\bar{S}^{(\delta_*)} = \{\bar{S}^{(\delta_*)}(t), t \in [0, T]\}$ is given by the equation

$$\bar{S}^{(\delta_*)}(t) = (Z(t))^{\frac{\nu}{2}-1}, \tag{3.1}$$

see (2.17). Here $Z = \{Z(t), t \in [0, T]\}$ is a time transformed squared Bessel process of dimension $\nu > 2$ and satisfies the SDE

$$dZ(t) = \frac{\nu}{4}\gamma(t)\, dt + \sqrt{\gamma(t)Z(t)}\, d\hat{W}(t) \tag{3.2}$$

for $t \in [0, T]$. Here the *scaling factor* $\gamma = \{\gamma(t), t \in [0, T]\}$ is an adapted stochastic process that will be specified later on.

By application of the Itô formula together with (3.1) and (3.2) for the benchmarked savings account

$$\hat{S}^{(0)}(t) = \frac{S^{(0)}(t)}{S^{(\delta_*)}(t)} \tag{3.3}$$

we see that

$$d\hat{S}^{(0)}(t) = d((Z(t))^{1-\frac{\nu}{2}}) = \left(1 - \frac{\nu}{2}\right)\sqrt{\gamma(t)}Z(t)^{\frac{1}{2}-\frac{\nu}{2}}\, d\hat{W}(t) \tag{3.4}$$

for $t \in [0, T]$. Using the Itô formula together with (2.1), (3.1) and (3.2), the SDE for the GOP $S^{(\delta_*)}(t)$ is given by

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t)\Bigg[\left(r(t) + \left(\frac{\nu}{2} - 1\right)^2 \gamma(t)\left(\frac{S^{(\delta_*)}(t)}{S^{(0)}(t)}\right)^{\frac{2}{2-\nu}}\right)dt + \left(\frac{\nu}{2} - 1\right)\sqrt{\gamma(t)}\left(\frac{S^{(\delta_*)}(t)}{S^{(0)}(t)}\right)^{\frac{1}{2-\nu}} d\hat{W}(t)\Bigg] \tag{3.5}$$

for $t \in [0, T]$. The SDE (3.5) is useful for certain types of derivative security valuation problems that are formulated using path dependent properties of the underlying GOP process $S^{(\delta_*)}$. By comparing (3.5) and (2.22) it follows that the discounted GOP drift is of the form

$$\alpha(t) = \left(\frac{\nu}{2} - 1\right)^2 \gamma(t)(Z(t))^{\frac{\nu}{2}-2} \tag{3.6}$$

for $t \in [0, T]$. Note that for dimension $\nu = 4$ the discounted GOP drift does not depend on Z.

3.2. Log-normal scaling

Since traders are very familiar with the log-normal process that drives the Black–Scholes dynamics let us assume that the scaling $\gamma = \{\gamma(t), t \in [0, T]\}$ follows a log-normal process that satisfies the SDE

$$\mathrm{d}\gamma(t) = \gamma(t)\big[\mu(t)\,\mathrm{d}t + \beta(t)\big(\varrho(t)\,\mathrm{d}\hat{W}(t) + \sqrt{1-\varrho(t)^2}\,\mathrm{d}\tilde{W}(t)\big)\big] \tag{3.7}$$

for $t \in [0, T]$. Here $\tilde{W}$ is a Wiener process that is independent of $W^1, \ldots, W^d$ and therefore $\hat{W}$. The scaling appreciation rate $\mu = \{\mu(t), t \in [0, T]\}$, the scaling correlation $\varrho = \{\varrho(t), t \in [0, T]\}$ and the scaling volatility $\beta = \{\beta(t), t \in [0, T]\}$ are assumed to be given deterministic functions of time.

This formulation for the dynamics of the scaling is chosen mainly for tractability reasons because, as mentioned above, log-normal dynamics are well-understood by practitioners and because it produces a natural growth behaviour that should be expected for the discounted GOP drift.

For this two-factor model the main stochastic volatility effect is produced by the squared Bessel process Z of dimension ν that is used to model the discounted GOP, see (3.1). However, derivative security prices and corresponding implied volatilities are also influenced by the stochastic properties of the discounted GOP drift. For example, we will see later on that higher scaling volatilities typically produce larger curvatures for the implied volatilities of European options. In the case when the dimension of ν does not equal four, then by (3.6) there is a correlation effect produced between the discounted GOP drift and the GOP.

3.3. Zero coupon bonds

Let us consider one of the simplest examples of a derivative security, namely a zero coupon bond that pays one unit of the domestic currency at the maturity date $\bar{T} \in [0, T]$. According to (2.15) and using (3.1), the fair benchmarked price $\hat{P}_{\bar{T}}(t, Z(t), \gamma(t))$ for a zero coupon bond is given by the formula

$$\hat{P}_{\bar{T}}(t, Z(t), \gamma(t)) = E\left(\frac{1}{S^{(\delta_*)}(\bar{T})}\middle|\mathcal{A}_t\right) = E\left(\frac{1}{S^{(0)}(\bar{T})(Z(\bar{T}))^{\frac{\nu}{2}-1}}\middle|\mathcal{A}_t\right) \tag{3.8}$$

for $t \in [0, \bar{T}]$. The corresponding price $P_{\bar{T}}(t, Z(t), \gamma(t))$ in domestic currency is then

$$P_{\bar{T}}(t, Z(t), \gamma(t)) = S^{(0)}(t)(Z(t))^{\frac{\nu}{2}-1}\hat{P}_{\bar{T}}(t, Z(t), \gamma(t)) \tag{3.9}$$

for $t \in [0, \bar{T}]$. For simplicity we will assume that the short rate process r is deterministic. Using (3.2) and (3.7) the benchmarked pricing function $\hat{P}_{\bar{T}}(\cdot, \cdot, \cdot)$ satisfies the Kolmogorov backward equation

$$\mathcal{L}^0 \hat{P}_{\bar{T}}(t, Z, \gamma) = 0 \tag{3.10}$$

for $(t, Z, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$ with boundary condition

$$\hat{P}_{\bar{T}}(\bar{T}, Z, \gamma) = \frac{1}{S^{(0)}(\bar{T})Z^{\frac{\nu}{2}-1}} \tag{3.11}$$

for $(Z, \gamma) \in (0, \infty)^2$. For this partial differential equation (PDE) the operator $\mathcal{L}^0$ when applied to a sufficiently smooth function $f : (0, \bar{T}) \times (0, \infty)^2 \to \Re$ is given by

$$\mathcal{L}^0 f(t, Z, \gamma) = \left[\frac{\partial}{\partial t} + \frac{\nu\gamma}{4}\frac{\partial}{\partial Z} + \gamma\mu(t)\frac{\partial}{\partial \gamma} + \frac{1}{2}\gamma Z\frac{\partial^2}{\partial Z^2} + \varrho(t)\beta(t)\gamma^{\frac{3}{2}}Z^{\frac{1}{2}}\frac{\partial^2}{\partial Z\partial\gamma} + \frac{1}{2}\beta(t)^2\gamma^2\frac{\partial^2}{\partial\gamma^2}\right]f(t, Z, \gamma) \tag{3.12}$$

for $(t, Z, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$.

As explained in section 3.1, the benchmarked savings account process $\hat{S}^{(0)} = \{\hat{S}^{(0)}(t), t \in [0, T]\}$, see (3.3) and (3.4), is an $(\underline{\mathcal{A}}, P)$-local martingale. Since it is non-negative it is an $(\underline{\mathcal{A}}, P)$-supermartingale, see Karatzas and Shreve (1991). This means that

$$\hat{S}^{(0)}(t) \geqslant E(\hat{S}^{(0)}(\bar{T})\big|\mathcal{A}_t)$$

for $t \in [0, \bar{T}]$, $\bar{T} \in [0, T]$. Consequently, combining (3.1) and (3.9) it is apparent that

$$P_{\bar{T}}(t, Z(t), \gamma(t)) \leqslant \frac{S^{(0)}(t)}{S^{(0)}(\bar{T})} \tag{3.13}$$

for $t \in [0, \bar{T}]$. For the above two-factor model figure 1 shows the difference $\frac{1}{S^{(0)}(\bar{T})} - P_{\bar{T}}(0, Z(0), \gamma(0))$ between the inverted deterministic savings account and the fair zero coupon bond as a function of the maturity date $\bar{T}$ and initial value $Z(0)$. For this and subsequent plots the default parameter values used were: $\nu = 4$, $r(t) = 0.04$, $\mu(t) = 0.04$, $\beta(t) = 1.0$ and $\varrho(t) = 0$ with initial values $Z(0) = 1.0$ and $\gamma(0) = 0.04$. Note that the quantities displayed in figure 1 are positive as is expected from (3.13). For initial values $Z(0)$ close to 1 and maturities less than two years the differences are very close to zero. These results together with those described in the remaining part of this paper were obtained using numerical PDE methods. The fact that the differences shown in figure 1 are not zero demonstrates that the process $\hat{S}^{(0)}$, see (3.4), is a strict $(\underline{\mathcal{A}}, P)$-local martingale. This conclusion also follows from results described in Revuz and Yor (1999).

This important observation means that the Radon–Nikodym derivative process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^{(0)}(t)}{\hat{S}^{(0)}(0)}, t \in [0, T]\}$ for what would be the risk neutral pricing measure Q, where $\frac{\mathrm{d}Q}{\mathrm{d}P}|_{\mathcal{A}_T} = \Lambda(T)$, is *not* an $(\underline{\mathcal{A}}, P)$-martingale. Consequently, for this model there is no equivalent risk neutral martingale measure. For additional commentary on these issues and to see why the benchmark approach is more general than the risk neutral approach we refer to Platen (2002).

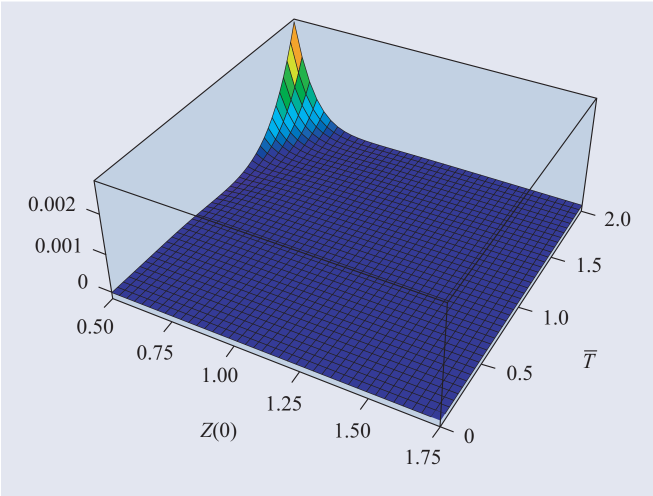

Figure 1. Difference between the inverted savings account and fair bond.

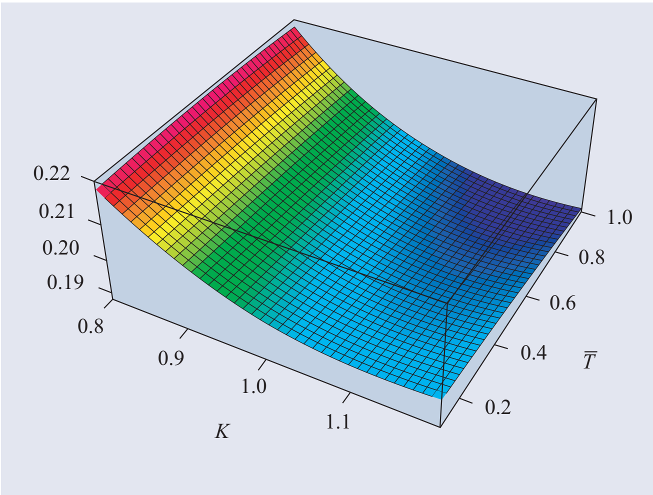

Figure 2. Implied volatilities for call options as a function of K and $\bar{T}$.

3.4. European options

Consider a European call option on the GOP $S^{(\delta_*)}$ with strike K and maturity date $\bar{T} \in [0, T]$. Using the benchmarked fair pricing formula (2.15) the benchmarked price $\hat{c}_{\bar{T},K}(t, Z(t), \gamma(t))$ for this contingent claim is then

$$\hat{c}_{\bar{T},K}(t, Z(t), \gamma(t)) = E\left(\left(1 - \frac{K}{S^{(0)}(\bar{T})Z(\bar{T})^{\frac{\nu}{2}-1}}\right)^+ \middle| \mathcal{A}_t\right) \tag{3.14}$$

for $t \in [0, \bar{T}]$. The corresponding option price $c_{\bar{T},K}(t, Z(t), \gamma(t))$ in domestic currency is obtained from (2.16) and is given by

$$c_{\bar{T},K}(t, Z(t), \gamma(t)) = S^{(0)}(t)(Z(t))^{\frac{\nu}{2}-1}\hat{c}_{\bar{T},K}(t, Z(t), \gamma(t)) \tag{3.15}$$

for $t \in [0, \bar{T}]$. Because of the form of (3.2) and (3.7) the benchmarked pricing function $\hat{c}_{\bar{T},K}(\cdot, \cdot, \cdot)$ satisfies the PDE

$$\mathcal{L}^0 \hat{c}_{\bar{T},K}(t, Z, \gamma) = 0 \tag{3.16}$$

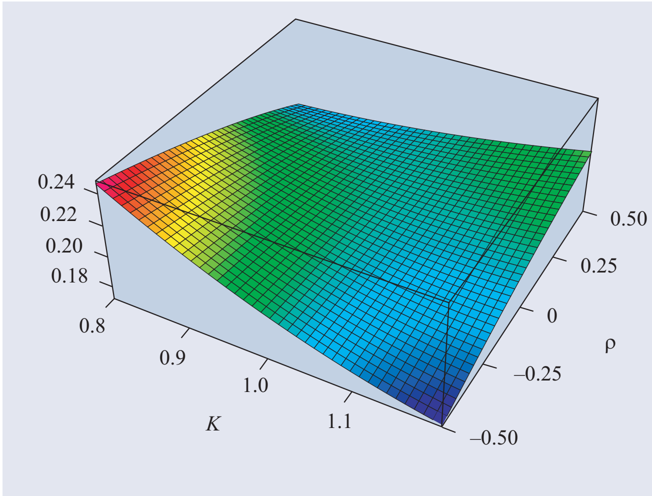

Figure 3. Implied volatilities for call options as a function of K and ϱ.

for $(t, Z, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$ with boundary condition

$$\hat{c}_{\bar{T},K}(\bar{T}, Z, \gamma) = \left(1 - \frac{K}{S^{(0)}(\bar{T})Z^{\frac{\nu}{2}-1}}\right)^+ \tag{3.17}$$

for $(Z, \gamma) \in (0, \infty)^2$.

To see the type of implied volatility curves that are produced by the MMM with log-normal scaling figure 2 displays a term structure of implied volatilities for European calls as a function of the maturity date $\bar{T}$ and strike K. These results were obtained by using the fair zero coupon bond price, see (3.9), to infer the discount factor used in the Black–Scholes formula. The implied volatilities shown in figure 2 are very close to those observed for European index options in real markets as, for example, shown in Cont and da Fonseca (2002). In particular, the at-the-money movements of the implied volatility surface are captured by the log-normal scaling.

Our formulation of the MMM with log-normal scaling allows for the Wiener processes driving the components $Z(t)$ and $\gamma(t)$ to be correlated. Figure 3 shows implied volatilities for European calls as a function of the strike K and correlation $\varrho(t) = \varrho$ for a fixed maturity date $\bar{T} = 0.25$. Note that by increasing ϱ the slope of the implied volatility curve as a function of the strike K also increases. A strong negative correlation produces a strongly negatively skewed implied volatility curve whereas a strong positive correlation generates a strongly positively skewed implied volatility curve.

To demonstrate the effect of making the scaling process γ stochastic we show in figure 4 implied volatilities for European calls as a function of the strike K and scaling volatility $\beta(t) = \beta$. The choice $\beta(t) = 0$ for $t \in [0, T]$ means that the scaling process γ will be deterministic. Figure 4 indicates that an increase in the scaling volatility $\beta(t) = \beta$ increases the curvature of the implied volatility curve viewed as a function of the strike K. By allowing for time dependent scaling volatilities this curvature can be controlled for different maturities.

The fair pricing formulae (2.16) can also be used to compute the fair price of a European put option. Thus, for

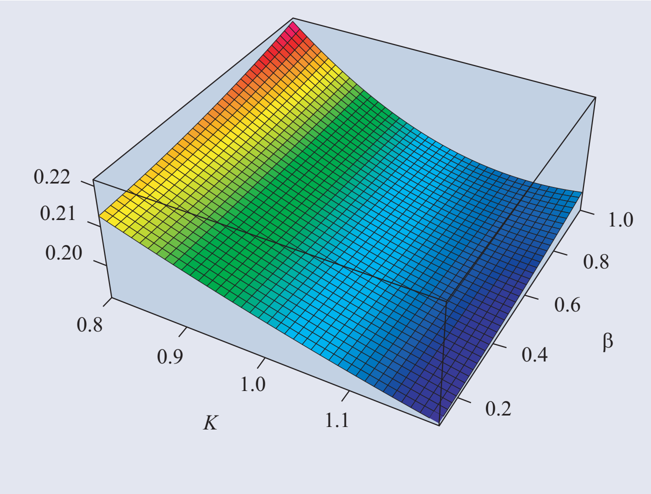

Figure 4. Implied volatilities for call options as a function of K and β.

a European put option on the GOP with strike K and maturity date $\bar{T} \in [0, T]$ the benchmarked fair price $\hat{p}_{\bar{T},K}(t, Z(t), \gamma(t))$ is given by

$$\hat{p}_{\bar{T},K}(t, Z(t), \gamma(t)) = E\left(\left(\frac{K}{S^{(0)}(\bar{T})(Z(\bar{T}))^{\frac{\nu}{2}-1}} - 1\right)^{+}\Bigg|\mathcal{A}_t\right) \tag{3.18}$$

for $t \in [0, \bar{T}]$. The corresponding option price in domestic currency, see (2.1), takes the form

$$p_{\bar{T},K}(t, Z(t), \gamma(t)) = S^{(0)}(t)(Z(t))^{\frac{\nu}{2}-1}\hat{p}_{\bar{T},K}(t, Z(t), \gamma(t)) \tag{3.19}$$

for $t \in [0, \bar{T}]$. Using (3.9), (3.15) and (3.19) it can be shown that

$$p_{\bar{T},K}(t, Z(t), \gamma(t)) = c_{\bar{T},K}(t, Z(t), \gamma(t)) - S^{(0)}(t)(Z(t))^{\frac{\nu}{2}-1} + K P_{\bar{T}}(t, Z(t), \gamma(t)) \tag{3.20}$$

for $t \in [0, \bar{T}]$. This is the put–call parity relation established for fair European option prices. By using this result it is evident that if the fair price of a zero coupon bond is used as the discount factor in the Black–Scholes formula, then the same implied volatilities will be returned for both European put and call options using the benchmark framework.

3.5. Binary options

As an example of an important class of path-dependent contingent claims we now consider the pricing of up-and-out and down-and-out binary options on the GOP under the MMM with log-normal scaling.

For maturity $\bar{T} \in (0, T]$ and levels $U > S^{(\delta_*)}(t)$ and $L < S^{(\delta_*)}(t)$ let τ^U and τ_L be stopping times given by

$$\tau^U = \inf\{s \geqslant 0 : (s, S^{(\delta_*)}(s)) \notin [0, \bar{T}) \times (0, U)\} \tag{3.21}$$

and

$$\tau_L = \inf\{s \geqslant 0 : (s, S^{(\delta_*)}(s)) \notin [0, \bar{T}) \times (L, \infty)\}. \tag{3.22}$$

Using the fair pricing formula (2.16) the benchmarked fair price for an up-and-out binary option with level U is then

$$\widehat{\mathrm{bin}}_{\mathrm{UO},\bar{T},U}(t, S^{(\delta_*)}(t), \gamma(t)) = E\left(\frac{\mathbf{1}_{\{\tau^U=\bar{T}\}}}{S^{(\delta_*)}(\bar{T})}\Bigg|\mathcal{A}_t\right) \tag{3.23}$$

for $t = [0, \bar{T}]$. The corresponding benchmarked fair price for a down-and-out binary option with level L is given by

$$\widehat{\mathrm{bin}}_{\mathrm{DO},\bar{T},L}(t, S^{(\delta_*)}(t), \gamma(t)) = E\left(\frac{\mathbf{1}_{\{\tau_L=\bar{T}\}}}{S^{(\delta_*)}(\bar{T})}\Bigg|\mathcal{A}_t\right) \tag{3.24}$$

for $t \in [0, \bar{T}]$. Note that because the definition of a binary option on the GOP relates to path-dependent properties of $S^{(\delta_*)}$ rather than the process Z, the pricing formulae (3.23) and (3.24) are expressed using the vector $(S^{(\delta_*)}, \gamma)$ rather than (Z, γ), as was used for the pricing of zero coupon bonds and European options.

Using (3.5) and (3.7) the benchmarked fair pricing function $\widehat{\mathrm{bin}}_{\mathrm{DO},\bar{T},L}(t, S^{(\delta_*)}, \gamma)$ satisfies the PDE

$$\mathcal{L}^0_*\widehat{\mathrm{bin}}_{\mathrm{DO},\bar{T},L}(t, S^{(\delta_*)}, \gamma) = 0 \tag{3.25}$$

for $(t, S^{(\delta_*)}, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$ with boundary conditions

$$\widehat{\mathrm{bin}}_{\mathrm{DO},\bar{T},L}(t, S^{(\delta_*)}, \gamma) = \frac{1}{S^{(\delta_*)}} \tag{3.26}$$

for $(S^{(\delta_*)}, \gamma) \in (0, \infty)^2$ and

$$\widehat{\mathrm{bin}}_{\mathrm{DO},\bar{T},L}(t, L, \gamma) = 0 \tag{3.27}$$

for $(t, \gamma) \in (0, \bar{T}) \times (0, \infty)$. Here $\mathcal{L}^0_*$ is the operator that applied to a sufficiently smooth function $f : [0, \bar{T}] \times (0, \infty)^2 \to \Re$ has the form

$$\begin{aligned}\mathcal{L}^0_* f(t, S^{(\delta_*)}, \gamma) = \Bigg[&\frac{\partial}{\partial t} + S^{(\delta_*)}(t) + \left\{r(t) + \left(\frac{\nu}{2} - 1\right)^2\gamma \times \left(\frac{S^{(\delta_*)}}{S^{(0)}(t)}\right)^{\frac{2}{2-\nu}}\right\}\frac{\partial}{\partial S^{(\delta_*)}} + \mu(t)\gamma\frac{\partial}{\partial\gamma} \\ &+ \frac{1}{2}\left(\frac{\nu}{2} - 1\right)^2\gamma(S^{(\delta_*)})^2\left(\frac{S^{(\delta_*)}}{S^{(0)}(t)}\right)^{\frac{2}{2-\nu}}\frac{\partial^2}{\partial(S^{(\delta_*)})^2} \\ &+ \varrho(t)\beta(t)\left(\frac{\nu}{2} - 1\right)S^{(\delta_*)}\gamma^{\frac{\nu}{2}}\left(\frac{S^{(\delta_*)}}{S^{(0)}(t)}\right)^{\frac{2}{2-\nu}}\frac{\partial^2}{\partial S^{(\delta_*)}\partial\gamma} \\ &+ \frac{1}{2}\beta(t)^2\gamma^2\frac{\partial^2}{\partial\gamma^2}\Bigg]f(t, S^{(\delta_*)}, \gamma)\end{aligned} \tag{3.28}$$

for $(t, S^{(\delta_*)}, \gamma) \in (0, T)\times(0, \infty)^2$. A similar PDE formulation holds for the up-and-out binary options.

To illustrate the effects of using the MMM with log-normal scaling for binary options we show in figure 5 price differences between this version of the MMM and corresponding Black–Scholes prices for down-and-out binary options on the GOP. These differences are shown for different values for the Black–Scholes binary option prices and the maturity date $\bar{T}$, obtained for a fixed scaling volatility $\beta(t) = 1.0$ and zero correlation $\varrho(t) = 0$.

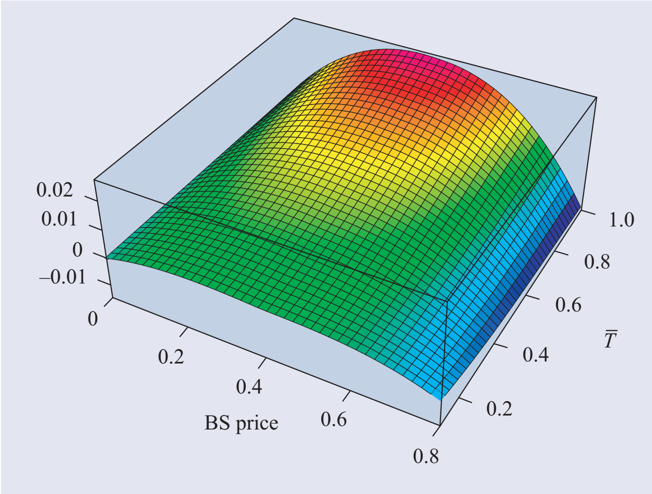

Figure 5. Differences between MMM and Black–Scholes prices for down-and-out binary options for different maturities $\bar{T}$.

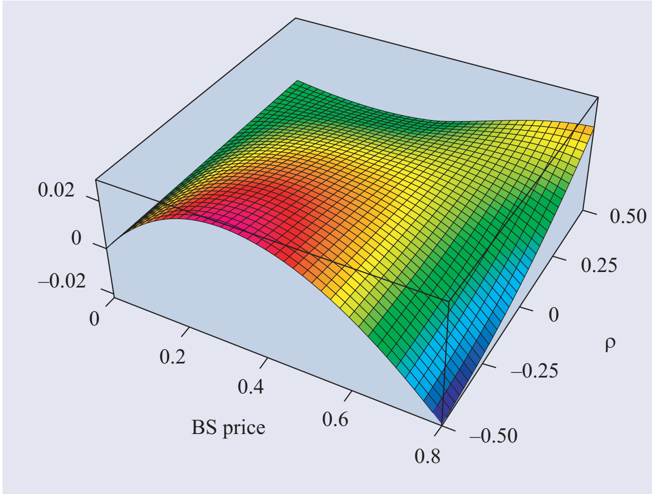

Figure 6. Differences between MMM and Black–Scholes prices for down-and-out binary options for different correlation ϱ.

We will now explain what is meant by corresponding Black–Scholes prices. To do this we first introduce the Black–Scholes down-and-out binary option price formula given by

$$\mathrm{bin}^{BS}_{DO,\bar{T},L}(t, S^{(\delta_*)}(t), \sigma) = e^{-r(\bar{T}-t)}\left(N(d_2(t)) - \left(\frac{L}{S^{(\delta_*)}(t)}\right)^{\frac{2r}{\sigma^2}-1} N(d_2(t))\right), \quad (3.29)$$

where

$$d_2(t) = \frac{\ln(\frac{S^{(\delta_*)}(t)}{L}) + (r - \frac{\sigma^2}{2})(\bar{T} - t)}{\sigma\sqrt{\bar{T} - t}}$$

for $t \in [0, \bar{T}]$, see Rubinstein and Reiner (1991). Here $N(\cdot)$ is the standard Gaussian distribution function and σ denotes the Black–Scholes volatility.

For all of the binary option examples described here the corresponding Black–Scholes binary option prices are obtained by choosing implied volatilities such that the prices for an at-the-money forward European call option for the MMM with log-normal scaling and the Black–Scholes model coincide. Here an at-the-money forward European call option is one that for $t = 0$ has a strike $\frac{K}{P_{\bar{T}}(0, Z(0), \gamma(0))}$ for fixed $K = 1.0$.

Different Black–Scholes binary option prices are obtained by varying the level L with $L < S^{(\delta_*)}(0)$ and maturity date $\bar{T}$ for fixed model parameters and initial values $S^{(\delta_*)}(0) = 1.0$ and $\gamma(0) = 0.04$. Figure 5 displays the typically shaped curves for down-and-out binary options observed in real markets. It can be seen that for a fixed scaling volatility $\beta(t) = 1.0$ the price differences between the MMM and corresponding Black–Scholes prices become smaller as the time to maturity $\bar{T}$ shortens. A similar type of plot can be obtained for up-and-out binary options. For up-and-in and down-and-in binary options hump-shaped price differences are also obtained. However, for these two types of securities the Black–Scholes prices are generally higher than the corresponding MMM prices and therefore the price differences are negative.

It is important to study the impact of non-zero correlation between the index $S^{(\delta_*)}(t)$ and the scaling factor $\gamma(t)$. For this purpose we show in figure 6 the price differences between the MMM with log-normal scaling and corresponding prices for Black–Scholes down-and-out binary options as a function of the Black–Scholes binary option price and correlation $\varrho(t) = \varrho$. These results were produced using a fixed maturity date $\bar{T} = 0.5$ and scaling volatility $\beta(t) = 1.0$. This figure shows that for $\varrho < 0$ decreasing the correlation increases the magnitude of the hump-shaped price difference curve. For $\varrho = 0.5$ the hump-shaped price difference curve becomes inverted to some degree. Finally, we display in figure 7 price differences between the MMM with log-normal scaling and corresponding Black–Scholes prices for down-and-out binary options as a function of the Black–Scholes binary option price and scaling volatility $\beta(t) = \beta$ with a fixed maturity date $\bar{T} = 0.5$ and zero correlation $\varrho(t) = 0$. Inspection of figure 7 demonstrates that, in general, increasing the scaling volatility β increases the price differences. Note also that for $\beta(t) = 0$ positive price differences are still obtained. This shows that even for the MMM with deterministic scaling a typical hump-shaped price difference curve is obtained, although the

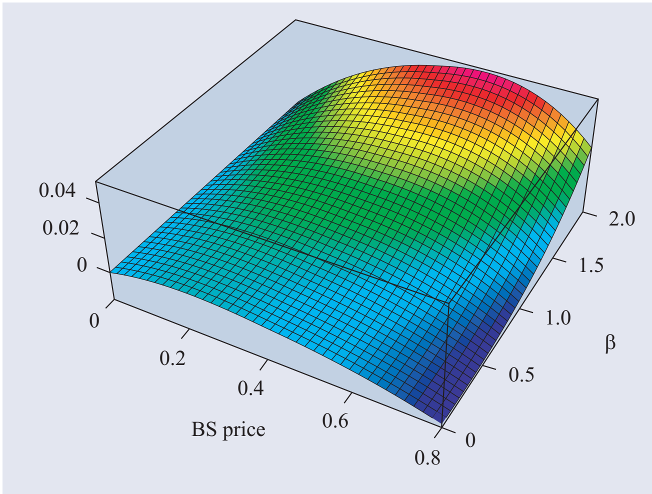

Figure 7. Differences between MMM and Black–Scholes prices for down-and-out binaries for different scaling volatility β.

magnitudes of the price differences are much smaller than for larger scaling volatilities. However, the MMM with log-normal scaling provides more realistic modelling capabilities and is better equipped to handle the dynamic movements of implied volatilities.

4. Conclusion

The MMM with log-normal scaling can be used to price a range of European and path-dependent contingent claims on a diversified index. Here the pricing system applied is based on the benchmark approach for which the reference unit chosen is the GOP. This pricing methodology is more general than risk neutral pricing. In fact, for the model under consideration there is no equivalent risk neutral martingale measure. Numerical results document the type of implied volatility term structures that are obtained for this two-factor model. Further computations show the price differences produced between the proposed model and corresponding Black–Scholes prices for a class of down-and-out binary options. These results generate patterns that are typically observed in real markets. Further research will focus on refining the model formulation for the stochastic scaling component and applying the pricing procedures for a wide class of path-dependent derivative securities.

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