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Abstract

Market efficiency tests that rely on the martingale difference behavior of returns can be based on various volatility measures. This paper argues that, to be able to differentiate between dependence and fat-tailedness. one should look simultaneously at plots based on absolute returns and variances. If the distribution is heavy-tailed, this shows up in the absolute moment plots, but not in the variance related plots. Linear dependence. by contrast, is revealed in both plots. We provide and discuss an analytical and a simulation experiment illustrating these points.

1 Introduction and Main Results

The efficiency of financial markets is a long-standing issue in the scientific literature. One way to test the efficient' market hypothesis for financial time series. e.g. stock prices and foreign exchange rates, is by checking the martingale difference behavior of the associated returns. This can be done. for example, using variance ratio tests as in Liu and He (1991), Lo and MacKinlay (1988, 1989), Poterba and Summers (1989), and Richardson and Stock (1989). Alternatively, one can use absolute moment ratio tests as put forward in Guillaume et al. (1994) and Müller et al. (1990), which are based on the mean absolute price change rather than the variance as a measure of volatility. Müller et al. (1990), using three years of intra-daily and fifteen years of daily data, present empirical evidence of a so-called scaling ***law,*** which relates the mean absolute value of the price change over a certain period to the size of the time interval in which the price change occurred.

The intuition behind the scaling law is quite straightforward. Consider the following simple random walk model,

$$p_t = p_{t-1} + \epsilon_t, \tag{1}$$

where p_t denotes the log price process, and where the disturbances ϵ_t are independent and identically distributed (i.i.d.). For simplicity, we consider the case where ϵ_t follows a normal distribution with mean zero and variance σ^2. Let r_t^n denote the n-period return at time t, i.e., $r_t^n \equiv p_t - p_{t-n}$. Using (1), we obtain by repeated substitution

$$r_t^n = \sum_{i=0}^{n-1} \epsilon_{t-i}. \tag{2}$$

Due to the independence of the innovations ϵ_t, the variance of r_t^n is equal to $n\sigma^2$. If we plot the logarithm of the variance of the n-period return against the logarithm of the return horizon, we find a linear function with slope equal to 1. We can thus test whether the random walk model is a valid description of the time series under investigation by checking whether the variance of the n-period return is linear in n with slope coefficient 1.

In this paper, we introduce a more generalized representation of the testing approach sketched above that simultaneously covers both the variance and the absolute variation as measures of volatility. Specifically, we propose to plot

$$\log\left(E\left(|r_t^n|^\zeta\right)\right) - \log\left(E\left(|r_t^1|^\zeta\right)\right) \tag{3}$$

versus the logarithm of ***n,*** with $0 \leqslant \zeta \leqslant 2$. Here E denotes the expectations operator. Note that by subtracting $\log(E(|r_t^1|^\zeta))$ from the n-period return measure, we force the plot to pass through the origin. For $\zeta = 2$, we obtain the variance related plot that was already discussed above, whereas for $\zeta = 1$ the plot is based on the expected absolute returns.

It can be shown (Groenendijk et al. (1997)) that, for the class of distributions which lie in the domain of attraction of a (symmetric) stable law, the relationship between (3) and log(n) is indeed asymptotically linear in *n*. Moreover, the slope of the corresponding linear function is equal to ζ/α, where $\alpha \in (0, 2]$ denotes the index of stability.[1] Consequently, we obtain a slope coefficient of 1 for variance related plots ($\zeta = 2$) and distributions that lie in the domain of attraction of the Gaussian distribution ($\alpha = 2$). For the expected absolute returns, by contrast, we obtain a line with slope coefficient 0.5 for finite variance innovations ϵ_t. The next section presents an analytical and a simulation experiment illustrating these findings.

2 Analytical and Monte Carlo Evidence

This section uses simple analytical derivations and Monte Carlo techniques to illustrate some of the points made in the previous section. Specifically, we provide plots of (3) against the logarithm of the return horizon for two stochastic processes for the one-period return series. First, we focus on a one-period return series that follows an autoregressive moving average (ARMA) process of order (1,1). Second, we consider one-period returns that are i.i.d. according to a fat-tailed Student-t distribution.

We first turn to the case where the one-period returns are linearly dependent. Consider the following stationary Gaussian ARMA(1,1) process for the one-period return series:

$$\begin{aligned} r_t^1 &= \varphi r_{t-1}^1 + \epsilon_t - \theta\epsilon_{t-1} \\ &= \varphi^t r_0^1 + \epsilon_t - \varphi^{t-1}\theta\epsilon_0 + \sum_{i=0}^{t-2}\varphi^i(\varphi-\theta)\epsilon_{t-1-i}, \quad |\varphi| < 1, \end{aligned} \tag{4}$$

where the ϵ_t are i.i.d. standard normal. Using straightforward substitution, one can derive that

$$\begin{aligned} r_{n-1}^n &= \sum_{t=0}^{n-1} r_t^1 \\ &= r_0^1\frac{1-\varphi^n}{1-\varphi} - \frac{1-\varphi^{n-1}}{1-\varphi}\theta\epsilon_0 + \sum_{t=1}^{n-1}\epsilon_t + (\varphi-\theta)\sum_{t=0}^{n-2}\sum_{i=0}^{t-1}\varphi^i\epsilon_{t-i} \\ &= r_0^1\frac{1-\varphi^n}{1-\varphi} - \frac{1-\varphi^{n-1}}{1-\varphi}\theta\epsilon_0 + \sum_{t=1}^{n-1}\epsilon_t + (\varphi-\theta)\sum_{t=1}^{n-2}\epsilon_t\frac{1-\varphi^{n-t-1}}{1-\varphi}. \end{aligned} \tag{5}$$

[1] Stable distributions are extensively discussed in Samorodnitsky and Taqqu (1994), which provides a comprehensive treatment; see also Ibragimov and Linnik (1971, ch. 2).

For the variance, this implies that for $n > 1$,

$$E\left(\left(r_{n-1}^{n}\right)^2\right) = \left(\frac{1-\varphi^n}{1-\varphi}\right)^2 \left(\frac{1-2\varphi\theta+\theta^2}{1-\varphi^2}\right) + \left(\frac{1-\varphi^{n-1}}{1-\varphi}\right)^2 \theta^2$$
$$- \frac{2\theta(1-\varphi^n)(1-\varphi^{n-1})}{(1-\varphi)^2} + 1 + \sum_{t=1}^{n-2}\left(1+(\varphi-\theta)\frac{1-\varphi^t}{1-\varphi}\right)^2 . \quad (6)$$

For n = 1, we obtain

$$E\left(\left(r_0^1\right)^2\right) = \frac{1-2\varphi\theta+\theta^2}{1-\varphi^2}. \quad (7)$$

For the AR(1) case, i.e., $\theta = 0$, (6) results in

$$\frac{E\left(\left(r_t^n\right)^2\right)}{E\left(\left(r_t^1\right)^2\right)} = \frac{1-\varphi^2}{(1-\varphi)^2} n - \frac{2\varphi(1-\varphi^n)}{(1-\varphi)^2}. \quad (8)$$

Note that for the absolute return case ($\zeta = 1$), we simply take the square root of the right-hand side of (8).

Equation (8) is plotted for several values of φ in Figures la-lb for $\zeta = 1$ and in Figures lc-ld for $\zeta = 2$. Note that Figures lb and 1d are plots of the deviations from the relevant reference line, which has slope 0.5 for $\zeta = 1$ and slope 1 for $\zeta = 2$, as explained in Section 1. For the case $\zeta = 1$, it appears that the dominating factor in (8) is of the order $\sqrt{n}$ for large n. Therefore, when plotting the logarithm of (3) as a function of the logarithm of the return horizon n, we expect a curve that converges to a linear function a + $b\log(n)$, with a = $2^{-1}\log((1-\varphi^2)/(1-\varphi)^2)$, and $b = 1/2$. This is clearly demonstrated in Figures la-lb. Apparently, not much is gained by considering the expected absolute return as a measure of volatility. The pattern of the plots for $\zeta = 2$ and $\zeta = 1$ are very similar, such that sticking to the traditional measure of volatility ($\zeta = 2$) seems adequate for linearly dependent one-period returns. We now turn to our second example, which shows in which cases the $\zeta = 1$ plots may contain useful additional information over the $\zeta = 2$ plots.

In our second example we consider Student-t distributions with $\nu = 2, \ldots, 7$ degrees of freedom. These distributions are fat-tailed, a characteristic that is shared by many empirical financial datasets, see, e.g., Campbell et al.' (1997) and de Vries (1994). Figures 2a-2d depict plots of Equation (3) versus log(n) for the Student-t distributions. For comparison, the normal distribution ($\nu = \infty$) is plotted as well. Figures 2a and 2b present graphs based on absolute returns ($\zeta = 1$), while Figures 2c and 2d are based on the variance as a volatility measure. Figure 2 is based on simulated data.

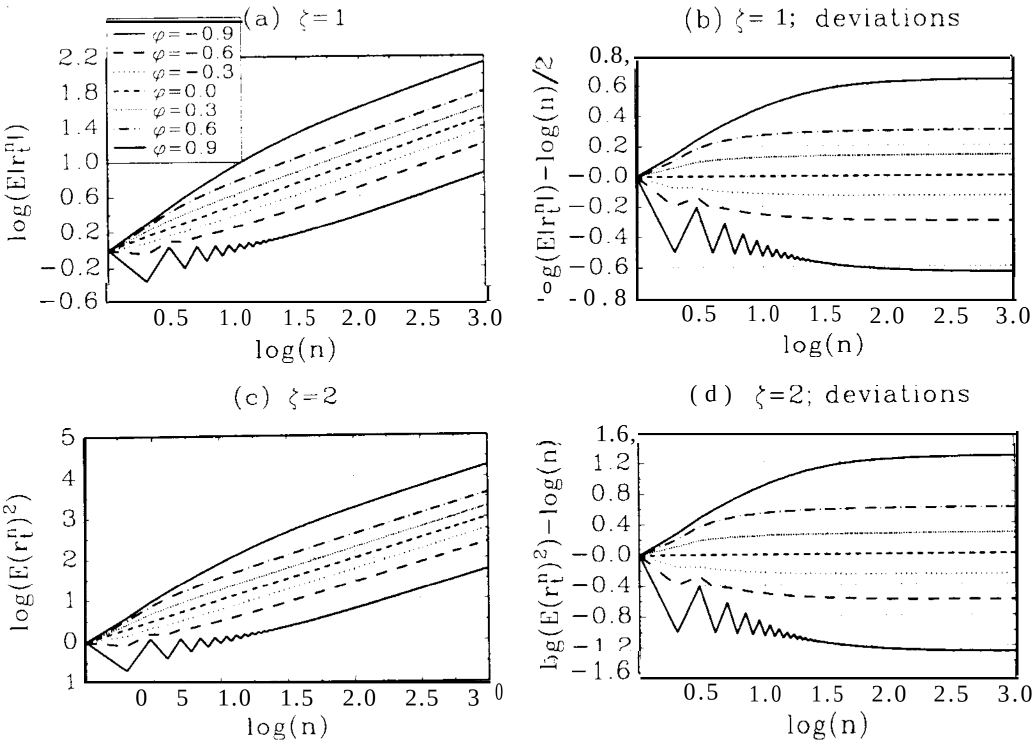

Figure 1: *Analytical absolute moment and variance ratio curves when returns are AR(1), with* φ *as indicated.*

The simulation experiment, is set up as follows. We draw 100 one-period return series of length 25 000 from the distribution under investigation. Then we compute the n-period return series from the one-period returns. with n ranging from 1 to 1000. For each of these series: we estimate the ζth absolute moment by averaging over the observed returns, i.e.. by using the estimator

$$\frac{1}{T-n+1}\sum_{t=n}^{T}|r_t^n|^\zeta, \tag{9}$$

with T = 25 000 denoting the sample size. Finally. we average over the 100 simulations performed. Note that since we use overlapping data to estimate the ζth power of the n-period returns. the number of observed returns ranges from 25 000 for n = 1 to 24001 for n = 1000. whereas. if we were to use non-overlapping data. the number of returns would be 25 000 and 25, respectively.

There are three notable features of Figure 2. The first is that for ζ = 1, the curves rise more steeply than the reference line ($\nu = \infty$), especially for low values of n. For large values of n. the curves become linear and parallel to the line through the origin with slope coefficient 0.5. The second notable

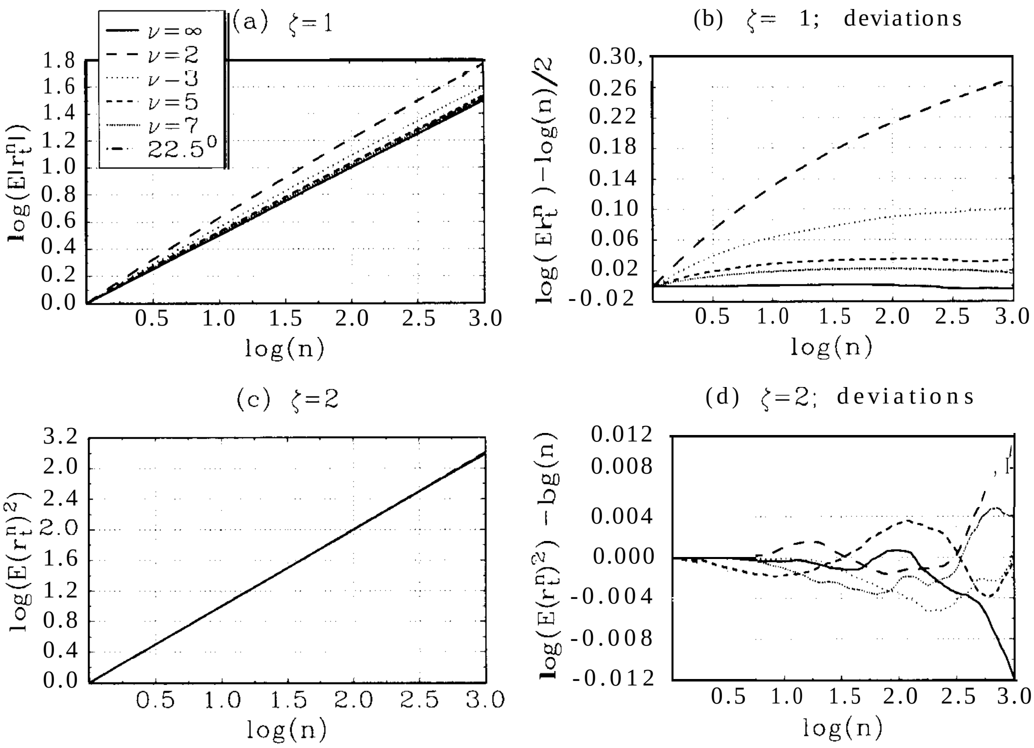

Figure 2: *Simulated absolute moment and variance ratio curves for Student-t(v) distributed ϵ_t, with ν as indicated.*

feature is that the level difference with the reference line is decreasing in the degrees of freedom parameter ν, i.e., the fatter the tails of the distribution, the larger the deviations from the 22.5 degrees line, and the more time it takes for the slopes of the scaling law plots to converge to their limiting value. The third is that almost no information is obtained from the variance ratio plots, which all lie on the 45 degrees line (the deviations plotted in Figure 2d are of negligible magnitude). This illustrates that it is not, informative to consider variance ratio plots when one is interested in the tail shape of the distribution, as is the case in, for instance, risk management decisions.

The results of expected absolute return related plots for the Student,-t distribution as given in Figure 2 have important, consequences for market, efficiency testing procedures which are based on these plots. The procedure that is usually followed for estimating the slope of the relevant, curve (3), fits a linear function of the form $f(n) = \beta \cdot \log(n)$ to this curve, with β some constant. If the curve one tries to fit is in reality nonlinear, as is the case with the Student-t distribution, the estimate of the (asymptotic) slope of the linear function will be generally biased. Specifically, the estimate of

β will be above the asymptotic slope coefficient of 0.5 if too many values of n near the origin are taken into account and the one-period return series is sufficiently fat-tailed. This illustrates that one should be very careful in concluding market inefficiency if the estimate of β departs from the reference value. since this result might just as well be due to nonnormality if too many low values of n are taken into account. Note that the scaling law plots for the Student-t distribution closely resemble the curves for positive values of the autoregressive parameter φ (Figure 1). This indicates that, it is difficult. if not impossible, to distinguish between certain forms of dependence and leptokurtosis when looking only at one of the two types of plots. This problem disappears if one considers both plots simultaneously.

3 Concluding Remarks

A popular way of testing the efficient market. hypothesis relies on the analysis of the martingale difference behavior of returns. Although the variance is by far the most commonly used measure of volatility, other dispersion measures. such as the absolute return have been used in the literature as well. This paper argues that, the most, informative testing procedure is to look simultaneously at plots based on both measures. If the returns exhibit some form of (linear) dependence. this shows up in both graphs. If the distribution of the return series has fat tails. this is revealed only in the absolute moment plot. Disentangling linear dependence and leptokurtosis is thus possible only by jointly considering absolute moments and variances.

References

Campbell. J.Y.. A.W. Lo. and A.C. MacKinlay (1997): *The econometrics of financial markets*. Princeton. NJ: Princeton University Press.

Groenendijk. P.A.. A. Lucas, and C.G. de Vries (1997): "A graphical generalized volatility ratio test with bootstrapped uniform confidence intervals," Mimeo. Free University Amsterdam.

Guillaume. D.M.. M.M. Dacorogna. R.R. Davé. U.A. Müller. R.B. Olsen, and O.V. Pictet (1994): "From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets," Discussion paper DMG 1994-04-06. Olsen & Associates.

Ibragimov. I.A.. and Yu V. Linnik (1971): *Independent and stationary sequences of random variables.* Groningen: Wolters-Noordhoff.

Liu, C.Y., and J. He (1991): “A variance-ratio test of random walks in foreign exchange rates,” *Journal of Finance, 46, 773-85.*

Lo, A.W., and A.C. MacKinlay (1988): “Stock market prices do not follow random walks: Evidence from a simple specification test,” *Review of Financial Studies,* 1, 41-66.

Lo, A.W., and A.C. MacKinlay (1989): “The size and power of the variance ratio test in finite samples: A Monte Carlo investigation,” *Journal of Econometrics, 40, 203-38.*

Müller, U.A., M.M. Dacorogna, R.B. Olsen, O.V. Pictet, M. Schwarz, and C. Morgenegg (1990) : ‘Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis,” *Journal of Banking and Finance,* 14, 1189-1208.

Poterba, J.M., and L.H. Summers (1988): “Mean reversion in stock prices: Evidence and implications,” *Journal of Financial Economics, 22, 27-59.*

Richardson, M., and J.H. Stock (1989): “Drawing inferences from statistics based on multiyear asset returns:” *Journal of Financial Economics, 25,* 323-48.

Samorodnitsky, G., and M.S. Taqqu (1994): *Stable non-Gaussian random processes.. Stochastic models with infinite variance.* New York: Chapman & Hall.

Vries, C.G. de (1994): “Stylized facts of nominal exchange rate returns,” in *The handbook of international macroeconomics*, ed. by F. van der Ploeg. Oxford, UK: Blackwell.