

## Scaling of the distribution of fluctuations of financial market indices

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We study the distribution of fluctuations of the S&P 500 index over a time scale  $\Delta t$  by analyzing three distinct databases. Database (i) contains approximately 1 200 000 records, sampled at 1-min intervals, for the 13-year period 1984–1996, database (ii) contains 8686 daily records for the 35-year period 1962–1996, and database (iii) contains 852 monthly records for the 71-year period 1926–1996. We compute the probability distributions of returns over a time scale  $\Delta t$ , where  $\Delta t$  varies approximately over a factor of  $10^4$ —from 1 min up to more than one month. We find that the distributions for  $\Delta t \leq 4$  d (1560 min) are consistent with a power-law asymptotic behavior, characterized by an exponent  $\alpha \approx 3$ , well outside the stable Lévy regime  $0 < \alpha < 2$ . To test the robustness of the S&P result, we perform a parallel analysis on two other financial market indices. Database (iv) contains 3560 daily records of the NIKKEI index for the 14-year period 1984–1997, and database (v) contains 4649 daily records of the Hang-Seng index for the 18-year period 1980–1997. We find estimates of  $\alpha$  consistent with those describing the distribution of S&P 500 daily returns. One possible reason for the scaling of these distributions is the long persistence of the autocorrelation function of the volatility. For time scales longer than  $(\Delta t)_\times \approx 4$  d, our results are consistent with a slow convergence to Gaussian behavior. [S1063-651X(99)11211-X]

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### I. INTRODUCTION AND BACKGROUND

The analysis of financial data by methods developed for physical systems has a long tradition [1–4], and has recently attracted the interest of physicists [5–28]. Among the reasons for this interest is the scientific challenge of understanding the dynamics of a strongly fluctuating complex system with a large number of interacting elements. In addition, it is possible that the experience gained by studying complex physical systems might yield new results in economics.

Financial markets are complex dynamical systems with many interacting elements that can be grouped into two categories: (i) the *traders*—such as individual investors, mutual funds, brokerage firms, and banks—and (ii) the *assets*—such as bonds, stocks, futures, and options. Interactions between these elements lead to transactions mediated by the stock exchange. The details of each transaction are recorded for later analysis. The dynamics of a financial market are difficult to understand not only because of the complexity of its internal elements but also because of the many intractable external factors acting on it, which may even differ from market to market. Remarkably, the statistical properties of certain observables appear to be similar for quite different markets [29–43], consistent with the possibility that there may exist “universal” results.

The most challenging difficulty in the study of a financial market is that the nature of the interactions between the different elements comprising the system is unknown, as is the way in which external factors affect it. Therefore, as a starting point, one may resort to empirical studies to help uncover the regularities or “empirical laws” that may govern financial markets.

The interactions between the different elements comprising financial markets generate many observables such as the transaction price, the share volume traded, the trading fre-

quency, and the values of market indices (Fig. 1). A number of studies investigated the time series of returns on varying time scales  $\Delta t$  in order to probe the nature of the stochastic process underlying it [10,13–15,29–43]. For a time series  $S(t)$  of prices or market index values, the return  $G(t) \equiv G_{\Delta t}(t)$  over a time scale  $\Delta t$  is defined as the forward change in the logarithm of  $S(t)$  [44],

$$G_{\Delta t}(t) \equiv \ln S(t + \Delta t) - \ln S(t). \quad (1)$$

For small changes in  $S(t)$ , the return  $G_{\Delta t}(t)$  is approximately the forward relative change,

$$G_{\Delta t}(t) \approx \frac{S(t + \Delta t) - S(t)}{S(t)}. \quad (2)$$

In 1900, Bachelier proposed the first model for the stochastic process of returns—an uncorrelated random walk with independent, identically distributed (i.i.d.) Gaussian random variables [1]. This model is natural if one considers the return over a time scale  $\Delta t$  to be the result of many independent “shocks,” which then lead by the central limit theorem to a Gaussian distribution of returns [1]. However, empirical studies [4,10,13–15,29–43] showed that the distribution of returns [45] has pronounced tails in striking contrast to that of a Gaussian. To illustrate this fact, we show in Fig. 2 the 10-min returns of the S&P 500 market index [46] for 1986 and 1987, and contrast it with a sequence of i.i.d. Gaussian random variables. Both are normalized to have unit variance. Clearly, large events are very frequent in the data, a fact largely underestimated by a Gaussian process. Despite this empirical fact, the Gaussian assumption for the distribution of returns is widely used in theoretical finance because of the simplifications it provides in analytical calculation;

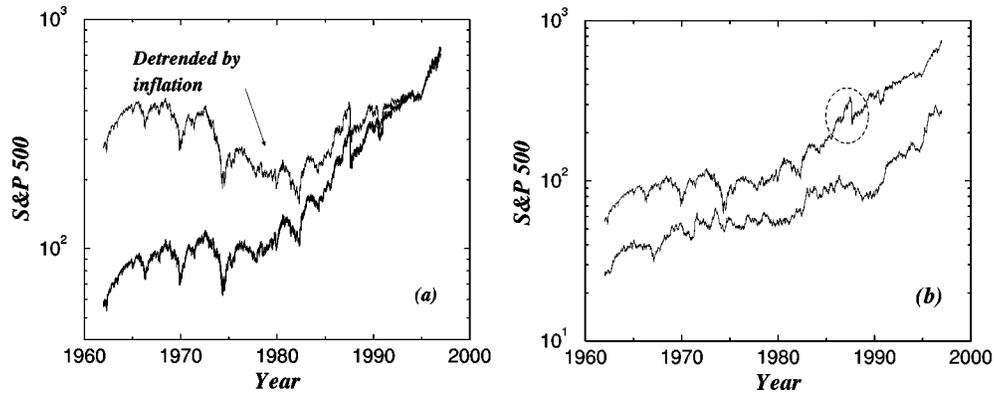


FIG. 1. The S&P 500 index is the sum of the market capitalizations of 500 companies. In (a), we display both the value of the S&P 500 index (bottom line) and the index detrended by inflation to 1994 U.S. dollars (top line). The sharp jump seen in 1987 is the market crash of October 19. (b) Comparison of the time evolution of the S&P 500 for the 35-year period 1962–1996 (top line) and a biased Gaussian random walk (bottom line). The random walk has the same bias as the S&P 500—approximately 7% per year for the period considered.

indeed, it is one of the assumptions used in the classic Black-Scholes option pricing formula [11,47–49].

In his pioneering analysis of cotton prices, Mandelbrot observed that in addition to being non-Gaussian, the process of returns shows another interesting property: “time scaling” — that is, the distributions of returns for various choices of  $\Delta t$ , ranging from 1 d up to one month have similar functional forms [4]. Motivated by (i) the pronounced tails, and (ii) the stable functional form for different time scales, Mandelbrot [4] proposed that the distribution of returns is consistent with a Lévy stable distribution [2,3] — that is, the returns can be modeled as a Lévy stable process. Lévy stable distributions arise from the generalization of the central limit theorem to random variables which do not have a finite second moment (see Appendix A).

Conclusive results on the distribution of returns are difficult to obtain, and require a large amount of data to study the rare events that give rise to the tails. More recently, the availability of high frequency data on financial market indices, and the advent of improved computing capabilities, has fa-

cilitated the probing of the asymptotic behavior of the distribution. For these reasons, recent empirical studies of the S&P 500 index such as Ref. [10] analyze typically  $10^6 - 10^7$  data points, in contrast to approximately 2000 data points analyzed in the classic work of Mandelbrot [4]. Reference [10] reports that the central part of the distribution of S&P 500 returns appears to be well fit by a Lévy distribution, but the asymptotic behavior of the distribution of returns shows faster decay than predicted by a Lévy distribution. Hence Ref. [10] proposed a truncated Lévy distribution—a Lévy distribution in the central part followed by an approximately exponential truncation—as a model for the distribution of returns. The exponential truncation ensures the existence of a finite second moment, and hence the truncated Lévy distribution is not a stable distribution [50,51]. The truncated Lévy process with i.i.d. random variables has slow convergence to Gaussian behavior due to the Lévy distribution in the center, which could explain the observed time scaling for a considerable range of time scales [10].

In addition to the probability distribution, a complementary aspect for the characterization of any stochastic process is the quantification of correlations. Studies of the autocorrelation function of returns show exponential decay with characteristic decay times  $\tau_{ch}$  of only 4 min [33,52–54]. As is clear from Fig. 3(a), for time scales beyond 20 min the correlation function is at the level of noise, in agreement with the *efficient market hypothesis* which states that it is not possible to predict future stock prices from their previous values [55]. If price correlations were not short range, one could devise a way to make money from the market indefinitely.

It is important to note that lack of linear correlation does not imply an i.i.d. process for the returns, since there may exist higher-order correlations [Fig. 3(b)]. Indeed, the amplitude of the returns, referred to in economics as the *volatility* [56], shows long-range time correlations that persist up to several months [14,33,53–63], and are characterized by an asymptotic power-law decay.

## II. MOTIVATION

A recent preliminary study reported that the distributions of 5-min returns for 1000 individual stocks and the S&P 500

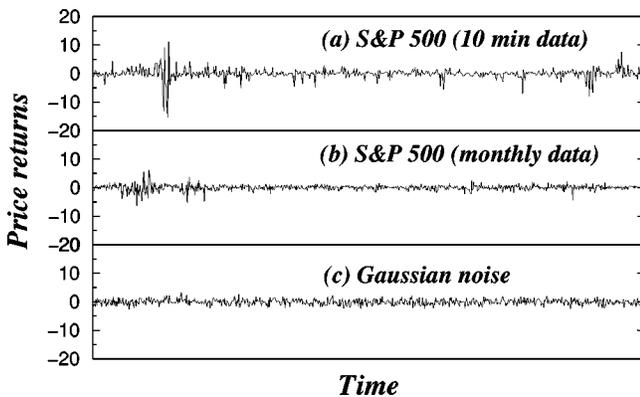


FIG. 2. Sequence of (a) 10-min returns, from database (i), and (b) one-month returns, from database (iii), for the S&P 500, normalized to unit variance. (c) Sequence of i.i.d. Gaussian random variables with unit variance, which was proposed by Bachelier as a model for stock returns [1]. For all three panels, there are 850 events —i.e., in panel (a) 850 min and in panel (b) 850 months. Note that, in contrast to (a) and (b), there are no “extreme” events in (c).

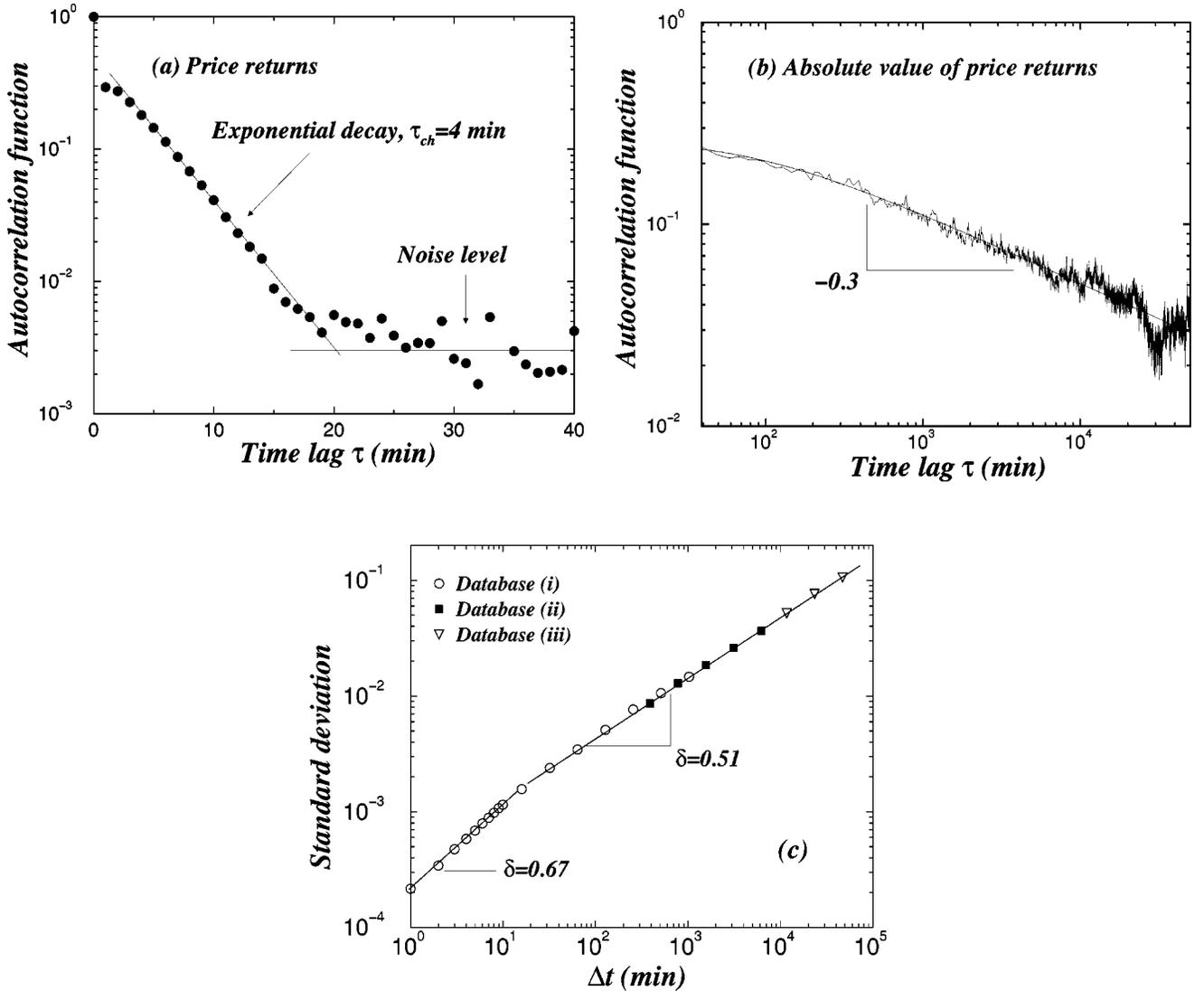


FIG. 3. (a) Semilog plot of the autocorrelation function for the S&P 500 returns  $G_{\Delta t}(t)$  sampled at a  $\Delta t=1$  min time scale,  $C_{\Delta t}(\tau) \equiv [\langle G_{\Delta t}(t) G_{\Delta t}(t+\tau) \rangle - \langle G_{\Delta t}(t) \rangle^2] / [\langle G_{\Delta t}(t)^2 \rangle - \langle G_{\Delta t}(t) \rangle^2]$ . The straight line corresponds to an exponential decay with a characteristic decay time  $\tau_{ch}=4$  min. Note that after 20 min the correlations are at the noise level. (b) Log-log plot of the autocorrelation function of the absolute returns. The solid line is a power-law regression fit over the entire range, which gives an estimate of the power-law exponent,  $\eta = 0.29 \pm 0.05$ . Better estimates of this exponent can be obtained from the power spectrum or from other more sophisticated methods. It has been recently reported using such methods that the autocorrelation function of the absolute value of the returns shows *two* power-law regimes with a crossover at approximately 1.5 days [58]. (c) Log-log plot of the time averaged volatility  $v \equiv v(\Delta t)$  as a function of the time scale  $\Delta t$  of the returns obtained from databases (i)–(iii). For  $\Delta t \leq 20$  min, we observe a slope  $\delta = 0.67 \pm 0.03$ , due to the exponentially damped time correlations. For  $\Delta t \geq 20$  min, we observe  $\delta = 0.51 \pm 0.06$ , indicating the absence of significant correlations.

index decay as a power law with an exponent  $\alpha \approx 3$ , well outside the stable Lévy regime [34]. Consistent results for the value of  $\alpha$  in the range  $2 \leq \alpha \leq 4$  were reported by previous studies both on stock markets [30,35–37] and on foreign exchange markets [38–42]. These results raise two important questions:

First, the distribution of returns has a finite second moment; thus we would expect it to converge to a Gaussian because of the central limit theorem. On the other hand, preliminary studies suggest the distributions of returns retain their power-law functional form for long time scales. So we can ask which of these two scenarios is correct? We find that the distributions of returns retain their functional form for time scales up to approximately 4 d, after which we find

results consistent with a slow convergence to Gaussian behavior [10,42,43].

Second, power-law distributions with  $2 \leq \alpha \leq 4$  are not stable distributions, but the distribution of returns retains its functional form for a range of time scales. It is then natural to ask how can this *scaling behavior* possibly arise? One possible explanation is the recently proposed exponentially truncated Lévy distribution [10,50,51]. However, the truncated Lévy process is constructed out of i.i.d. random variables, and hence is not consistent with the empirically observed long persistence in the autocorrelation function of the volatility of returns [33,54–62]. Moreover, our data support the possibility that the asymptotic nature of the distribution is a power law with an exponent outside the Lévy regime.

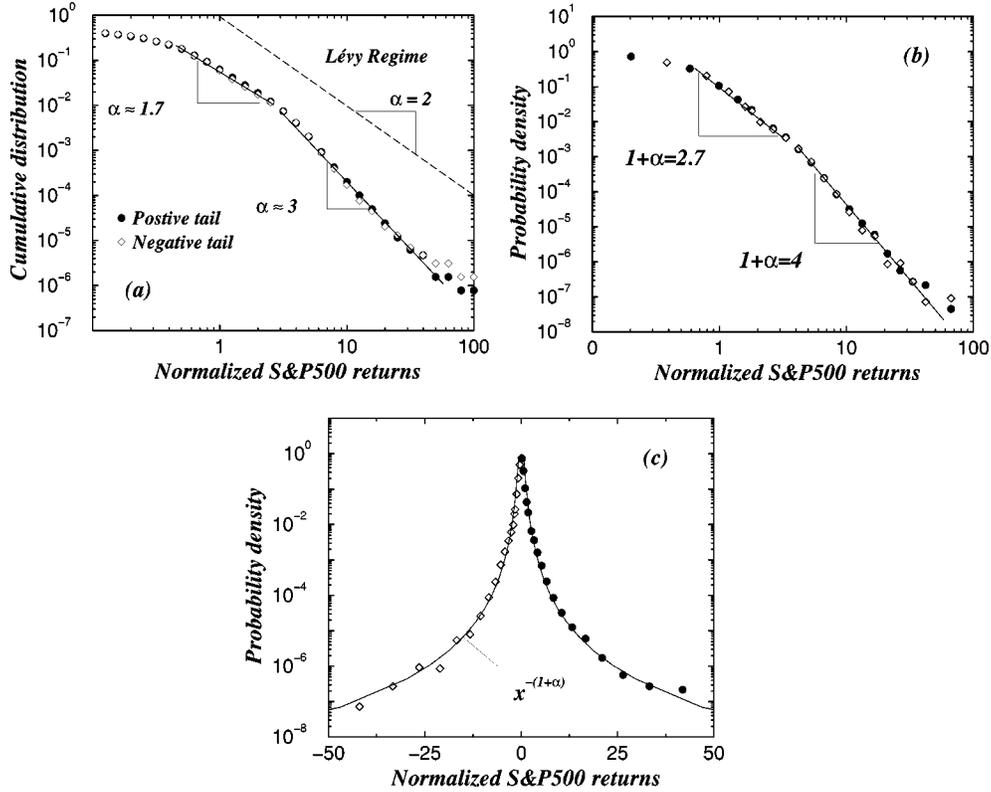


FIG. 4. (a) Log-log plot of the cumulative distribution of the normalized 1-min returns for the S&P 500 index. Power-law regression fits in the region  $3 \leq g \leq 50$  yield  $\alpha = 2.95 \pm 0.07$  (positive tail), and  $\alpha = 2.75 \pm 0.13$  (negative tail). For the region  $0.5 \leq g \leq 3$ , regression fits give  $\alpha = 1.6 \pm 0.1$  (positive tail), and  $\alpha = 1.7 \pm 0.1$  (negative tail). (b) Log-log and (c) linear-log plots of the probability density function for the normalized S&P 500 returns. The solid lines are power-law fits with exponents  $1 + \alpha \approx 4$ . Power-law regression fits in the region  $3 \leq g \leq 50$  yield estimates  $\alpha = 3.01 \pm 0.11$  (positive tail) and  $\alpha = 3.02 \pm 0.08$  (negative tail).

Also, we will argue that the scaling behavior observed in the distribution of returns may be connected to the slow decay of the volatility correlations.

The organization of the paper is as follows. Section III describes the data analyzed. Sections IV and V study the distribution of returns of the S&P 500 index on time scales  $\Delta t \leq 1$  d and  $\Delta t > 1$  d, respectively. Section VI discusses how time correlations in volatility are related to the time scaling of the distributions, and Sec. VII presents concluding remarks.

### III. DATA ANALYZED

We analyze the S&P 500 index, which comprises 500 companies chosen for market size, liquidity, and industry group representation in the US. The S&P 500 is a market-value weighted index (stock price times number of shares outstanding), with each stock's weight proportional to its market value. The S&P 500 index is one of the most widely used benchmarks of U.S. equity performance. In our study, we first analyze database (i) which contains ‘‘high-frequency’’ data that covers the 13 years period 1984–1996, with a recording frequency of less than 1 min. The total number of records in this database exceeds  $4.5 \times 10^6$ . To investigate longer time scales, we study two other databases. Database (ii) contains daily records of the S&P 500 index for the 35-year period 1962–1996, and database (iii) contains monthly records for the 71-year period 1926–1996.

In order to test if our results are limited to the S&P 500

index, we perform a parallel analysis on two other market indices. Database (iv) contains 3560 daily records of the NIKKEI index of the Tokyo stock exchange for the 14-year period 1984–1997, and database (v) contains 4649 daily records of the Hang-Seng index of the Hong Kong stock exchange for the 18-year period 1980–1997.

## IV. DISTRIBUTION OF RETURNS FOR $\Delta t \leq 1$ DAY

### A. Distribution of returns for $\Delta t = 1$ min

We first analyze the values of the S&P 500 index from the high-frequency data for the 13-year period 1984–1996, which extends the database studied in Ref. [10] by an additional seven years. The data are typically recorded at 15-s intervals. We first sample the data at 1-min intervals and generate a time series  $S(t)$  with approximately 1.200 000 data points. From the time series  $S(t)$ , we compute the return  $G \equiv G_{\Delta t}(t)$  which is the relative change in the index, defined in Eq. (1).

In order to compare the behavior of the distribution for different time scales  $\Delta t$ , we define a normalized return  $g \equiv g_{\Delta t}(t)$ ,

$$g \equiv \frac{G - \langle G \rangle_T}{v}. \quad (3)$$

Here the time averaged volatility  $v \equiv v(\Delta t)$  is defined through  $v^2 \equiv \langle G^2 \rangle_T - \langle G \rangle_T^2$ , and  $\langle \dots \rangle_T$  denotes an average

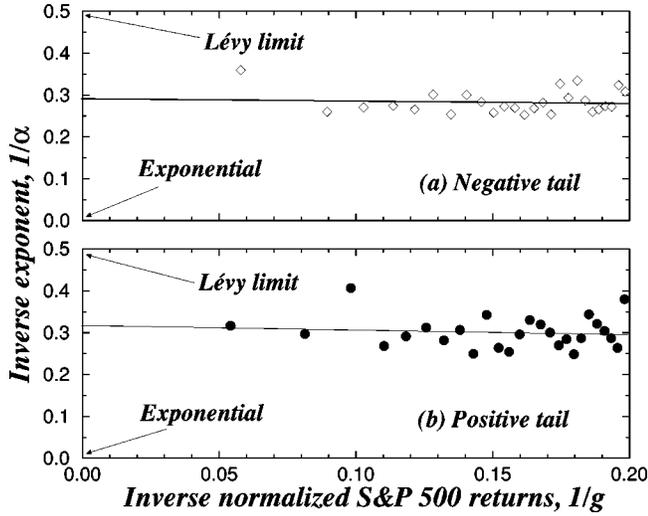


FIG. 5. Inverse local slopes of the cumulative distributions of normalized returns for  $\Delta t = 1$  min for the (a) positive and (b) negative tails. Each point is an average over 100 different inverse local slopes. Extrapolation of the regression lines provides estimates for the asymptotic slopes  $\alpha = 3.45 \pm 0.07$  (positive tail) and  $\alpha = 3.29 \pm 0.07$  (negative tail).

over the entire length of the time series. Figure 4(a) shows the cumulative distribution of returns for  $\Delta t = 1$  min. For both positive and negative tails, we find a power-law asymptotic behavior

$$P(g > x) \sim \frac{1}{x^\alpha}, \quad (4)$$

similar to what was found for individual stocks [34]. For the region  $3 \leq g \leq 50$ , regression fits yield

$$\alpha = \begin{cases} 2.95 \pm 0.07 & \text{(positive tail)} \\ 2.75 \pm 0.13 & \text{(negative tail),} \end{cases} \quad (5)$$

well outside the Lévy stable range,  $0 \leq \alpha < 2$ . Consistent values for  $\alpha$  are also obtained from the density function. For a more accurate estimation of the asymptotic behavior, we use the modified Hill estimator [Figs. 5(a) and 5(b); see also Appendix B]. We obtain estimates for the asymptotic slope in the region  $3 \leq g \leq 50$ :

$$\alpha = \begin{cases} 3.45 \pm 0.07 & \text{(positive tail)} \\ 3.29 \pm 0.07 & \text{(negative tail).} \end{cases} \quad (6)$$

For the region  $g \leq 3$ , regression fits yield smaller estimates of  $\alpha$ , consistent with the possibility of a Lévy distribution in the central region. The values of  $\alpha$  obtained in this range are quite sensitive to the bounds of the region used for fitting. Our estimates range from  $\alpha \approx 1.35$  up to  $\alpha \approx 1.8$  for different fitting regions in the interval  $0.1 \leq g \leq 6$ . For example, in the region  $0.5 \leq g \leq 3$ , we obtain

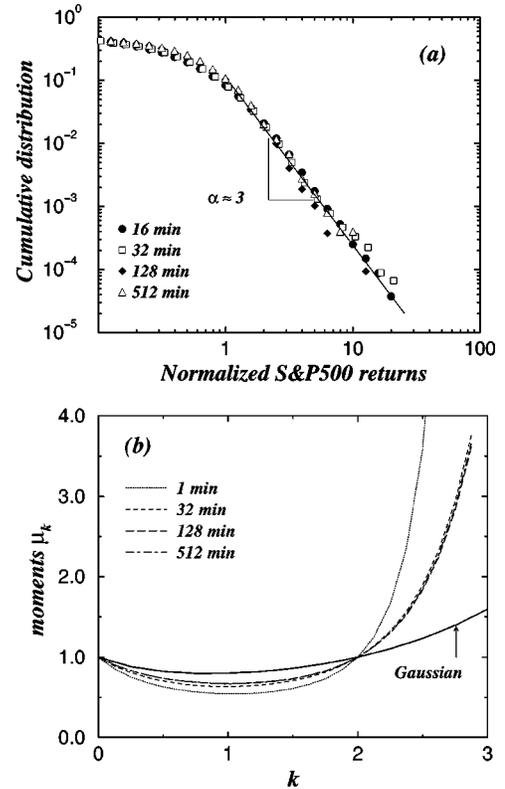


FIG. 6. (a) Log-log plot of the cumulative distribution of normalized returns of the positive tails for  $\Delta t = 16, 32, 128,$  and  $512$  min. Power-law regression fits yield estimates of the asymptotic power-law exponents  $\alpha = 2.69 \pm 0.04$ ,  $\alpha = 2.53 \pm 0.06$ ,  $\alpha = 2.83 \pm 0.18$ , and  $\alpha = 3.39 \pm 0.03$  for  $\Delta t = 16, 32, 128,$  and  $512$  min, respectively. (b) The moments of the distribution for  $\Delta t = 1, 32, 128,$  and  $512$  min. The change in the behavior of the moments from the 1-min scale is probably the effect of the gradual disappearance of the Lévy slope for small values of  $g$ . For  $\Delta t > 30$  min there is no region with slopes in the Lévy range, and we observe good agreement between all time scales.

$$\alpha \approx \begin{cases} 1.6 & \text{(positive tail)} \\ 1.7 & \text{(negative tail),} \end{cases} \quad (7)$$

which are consistent with the result  $\alpha \approx 1.4$  found for small values of  $g$  in Ref. [10]. Note that in Ref. [10] the estimates of  $\alpha$  were calculated using the scaling form of the return probability to the origin  $P(0)$ . It is possible that for the financial data analyzed here,  $P(0)$  is not the optimal statistic, because of the discreteness of the individual-company distributions that comprise it [64]. It is also possible that our values of  $\alpha$  for small values of  $g$  could be due to the discreteness in the returns of the individual companies comprising the S&P 500.

### B. Scaling of the distribution of returns for $\Delta t$ up to 1 d

Next, we study the distribution of normalized returns for longer time scales. Figure 6(a) shows the cumulative distribution of normalized S&P 500 returns for time scales up to 512 min (approximately 1.5 d). The distribution appears to retain its power-law functional form for these time scales. We verify this scaling behavior by analyzing the moments of the distribution of normalized returns  $g$ ,

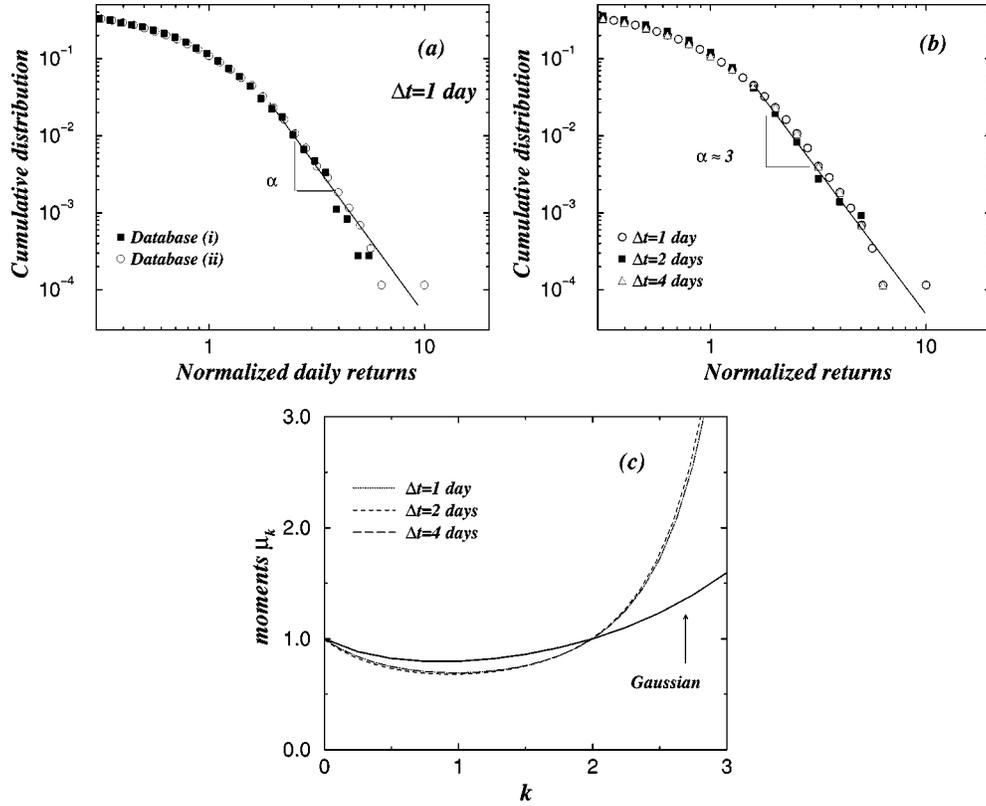


FIG. 7. (a) Cumulative distribution of the normalized S&P 500 daily returns from two different databases: database (i), which contains 1-min records for 13 years; and database (ii), which contains daily records for 35 years. Power-law regression fits in the region  $g \geq 1$  lead to the estimates  $\alpha = 3.75 \pm 0.30$  for database (i) and  $\alpha = 3.66 \pm 0.11$  for database (ii). (b) The cumulative distribution from database (ii) for  $\Delta t = 1, 2,$  and  $4$  d. The apparent scaling behavior of these distributions is confirmed by the estimates  $\alpha = 3.75 \pm 0.41$  ( $\Delta t = 2$  d) and  $\alpha = 3.77 \pm 0.29$  ( $\Delta t = 4$  d). (c) The behavior of the moments for these time scales is in agreement with the apparent scaling behavior.

$$\mu_k \equiv \langle |g|^k \rangle_T, \quad (8)$$

where  $\langle \dots \rangle_T$  denotes an average over all the normalized returns for all the bins. Since  $\alpha \approx 3$ , we expect  $\mu_k$  to diverge for  $k \geq 3$ , and hence we compute  $\mu_k$  for  $k < 3$ .

Figure 6(b) shows the moments of the normalized returns  $g$  for different time scales from 5 min up to 1 d. The moments do not vary significantly for the above time scales, confirming the scaling behavior of the distribution observed in Fig. 6(a).

## V. DISTRIBUTION OF RETURNS FOR $\Delta t \geq 1$ DAY

### A. S&P 500 index

For time scales beyond 1 d [66], we use database (ii) which contains daily sampled records of the S&P 500 index for the 35-year period 1962–1996. Figure 7(a) shows the agreement between distributions of normalized S&P 500 daily returns from database (i), which contains 1-min sampled data, and database (ii), which contains daily sampled data. Regression fits for the region  $1 \leq g \leq 10$  give estimates of  $\alpha \approx 3$ . Figure

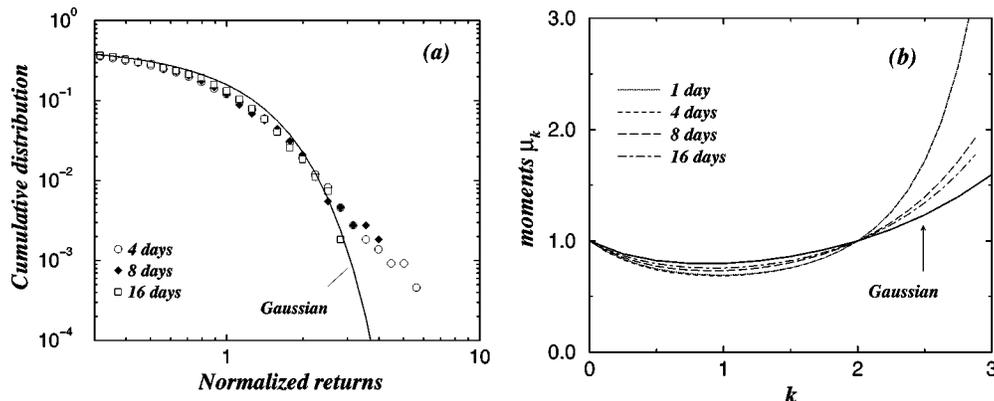


FIG. 8. (a) Cumulative distribution for the positive tail of S&P 500 returns for time scales  $\Delta t = 4, 8,$  and  $16$  d. The bold curve shows the cumulative distribution of a Gaussian with zero mean and unit variance. (b) The moments for time scales  $\Delta t = 8$  and  $16$  d are consistent with a slow convergence to Gaussian behavior. Note that the curves for  $\Delta t = 1$  and  $4$  d coincide.

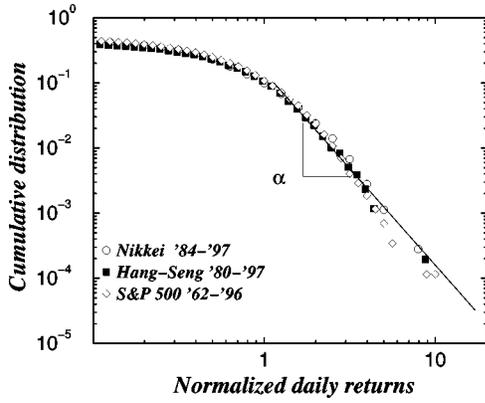


FIG. 9. Comparison of the cumulative distributions for the positive tails of the normalized returns for the daily records of the NIKKEI index from 1984 to 1997, the daily records of the Hang-Seng index from 1980 to 1997, and the daily records of the S&P 500 index. The apparent power-law behavior in the tails is characterized by the exponents  $\alpha=3.05\pm 0.16$  (NIKKEI),  $\alpha=3.03\pm 0.16$  (Hang-Seng), and  $\alpha=3.34\pm 0.12$  (S&P 500). The fits are performed in the region  $g \geq 1$ .

7(b) shows the scaling behavior of the distribution for  $\Delta t = 1, 2,$  and  $4$  d. For these choices of  $\Delta t$ , the scaling behavior is also visible for the moments [Fig. 7(c)].

Figure 8(a) shows the distribution of the S&P 500 returns for  $\Delta t=4, 8,$  and  $16$  d. The data are now consistent with a slow convergence to Gaussian behavior. This is also visible for the moments [Fig. 8(b)].

**B. NIKKEI and Hang-Seng indices**

The S&P 500 is but one of many stock market indices. Hence we investigate whether the above results regarding the power-law asymptotic behavior of the distribution of returns hold for other market indices as well. Figure 9 compares the distributions of daily returns for the NIKKEI index of the Tokyo stock exchange and the Hang-Seng index of the Hong Kong stock exchange with that of the S&P 500. The distributions have similar functional forms, suggesting the possi-

bility of “universal” behavior of these distributions. In addition, the estimates of  $\alpha$  from regression fits,

$$\alpha = \begin{cases} 3.05 \pm 0.16 & \text{(NIKKEI)} \\ 3.03 \pm 0.16 & \text{(Hang-Seng)}, \end{cases} \quad (9)$$

are in good agreement for the three cases.

**VI. DEPENDENCE OF AVERAGE VOLATILITY ON TIME SCALE**

The behavior of the time-averaged volatility  $v(\Delta t)$  as a function of the time scale  $\Delta t$  is shown in Fig. 3(c). We find a power-law dependence

$$v(\Delta t) \propto (\Delta t)^\delta. \quad (10)$$

We estimate  $\delta \approx 0.7$  for time scales  $\Delta t < 20$  min. This value is larger than  $1/2$  due to the exponentially damped time correlations, which are significant up to approximately 20 min. Beyond 20 min,  $\delta \approx 0.5$ , indicating the absence of correlations in the returns, in agreement with Fig. 3(a). The time-averaged volatility is also consistent with essentially uncorrelated behavior for the daily and monthly returns.

**VII. VOLATILITY CORRELATIONS AND TIME SCALING**

We have presented evidence that the distributions of returns retain the same functional form for a range of time scales (see Fig. 10 and Table I). Here, we investigate possible causes of this scaling behavior. Previous explanations of scaling relied on Lévy stable [4] and exponentially truncated Lévy processes [6,10]. However, the empirical data that we analyze are not consistent with either of these two processes.

**A. Rate of convergence**

We first compare the rate of convergence of the probability of the returns to that of a computer-generated time series which has the same distribution but is statistically independent by construction. This way, we will be able to study the

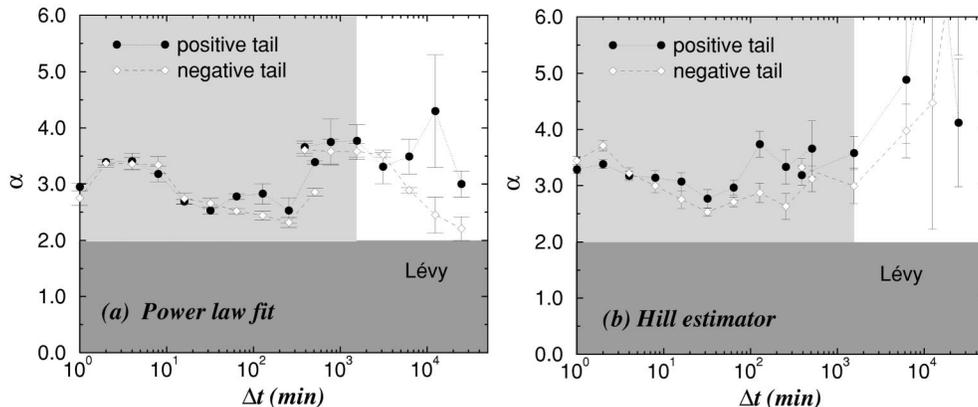


FIG. 10. The values of the exponent  $\alpha$  characterizing the asymptotic power-law behavior of the distribution of returns as a function of the time scale  $\Delta t$  obtained using (a) a power-law fit, and (b) the Hill estimator. The values of  $\alpha$  for  $\Delta t < 1$  d are calculated from database (i) which contains 13 years of 1-min records, while for  $\Delta t \geq 1$  d they are calculated from database (ii), which has 35 years of daily records. The unshaded region, corresponding to time scales larger than  $(\Delta t)_\times \approx 4$  d (1560 min), indicates the range of time scales where we find results consistent with slow convergence to Gaussian behavior (see the text and the preceding figures).

TABLE I. The values of the exponent  $\alpha$ , for different time scales  $\Delta t$ , for the S&P 500 index: (a) power-law regression fit to the cumulative distribution, and (b) Hill estimator. The daggered values are computed using database (ii), which contains daily-sampled records, while the values without the dagger are computed using database (i), which contains records with a 1-min sampling. Note that we use the conversion 1 d=390 min [65].

$\Delta t$ (min)	Power-law fit		Hill estimator	
	Positive	Negative	Positive	Negative
1	2.95±0.07	2.75±0.13	3.29±0.07	3.45±0.07
2	3.39±0.05	3.37±0.07	3.38±0.08	3.71±0.09
4	3.41±0.14	3.36±0.11	3.18±0.09	3.22±0.10
8	3.18±0.14	3.34±0.15	3.14±0.13	3.00±0.12
16	2.69±0.04	2.74±0.10	3.07±0.26	2.75±0.16
32	2.53±0.06	2.66±0.09	2.77±0.16	2.53±0.07
64	2.78±0.05	2.52±0.05	2.97±0.14	2.71±0.09
128	2.83±0.18	2.44±0.08	3.74±0.23	2.87±0.17
256	2.53±0.23	2.32±0.09	3.33±0.30	2.63±0.23
390 <sup>†</sup>	3.66±0.11	3.61±0.11	3.19±0.17	3.33±0.16
512	3.39±0.03	2.86±0.07	3.7±0.5	3.12±0.23
780 <sup>†</sup>	3.75±0.41	3.58±0.22	3.06±0.26	4.67±0.38
1560 <sup>†</sup>	3.77±0.29	3.58±0.14	3.58±0.29	2.99±0.32
3120 <sup>†</sup>	3.31±0.30	3.52±0.04	4.9±0.6	3.85±0.45
6240 <sup>†</sup>	3.49±0.31	2.89±0.05	4.9±1.1	3.97±0.48
12480 <sup>†</sup>	4.3±1.0	2.45±0.32	8.7±2.0	4.5±2.2
24960 <sup>†</sup>	3.00±0.23	2.21±0.21	4.1±1.1	7.7±2.4

convergence to Gaussian behavior of independent random variables distributed as a power law, with an exponent  $\alpha \approx 3$ .

Next we generate a time series  $X \equiv X_k$ ,  $k=1, \dots, 40 \times 10^6$  distributed as  $P(X > x) \sim 1/x^3$ . We next calculate the new random variables  $I_n \equiv \sum_{i=1}^n X_k$ , and compute the cumulative distributions of  $I_n$  for increasing values of  $n$ . These distributions show faster convergence with increasing  $n$  than the distributions of returns [Fig. 11(a)]. This convergence is also visible in the moments. Figures 11(a) and 11(b) show that for  $n=256$ , both the moments and the cumulative distribution show Gaussian behavior. In contrast, for the distribution of returns, we observe significantly slower convergence to Gaussian behavior: In the case of the S&P 500 index, one observes a possible onset of convergence for  $\Delta t \approx 4$  d (1560 min), starting from 1-min returns.

These results confirm the existence of time dependencies in the returns [33,53–61]. Next, we show that the scaling behavior observed for the S&P 500 index no longer holds when we destroy the dependencies between the returns at different times.

### B. Randomizing the time series of returns

We start with the 1-min returns and then destroy all the time dependencies that might be present by shuffling the time series of  $G_{\Delta t=1}(t)$ , thereby creating a new time series  $G_1^{\text{sh}}(t)$  which contains *statistically independent* returns. By adding up  $n$  consecutive returns of the shuffled series  $G_1^{\text{sh}}(t)$ , we construct the  $n$ -min returns  $G_n^{\text{sh}}(t)$ .

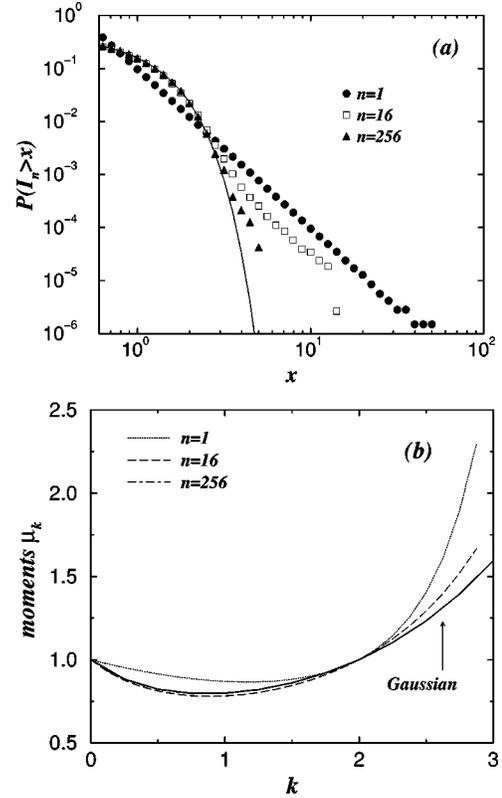


FIG. 11. Convergence of distribution for independent variables. We first generate a time series  $X_k$  distributed as  $P(X \geq x) \sim 1/x^3$ . We then generate the variables  $I_n \equiv \sum_{i=1}^n X_k$  for  $n=1, 16$ , and 256. (a) Cumulative distributions of  $I_n$ . Note that the curve for  $n=256$  is indistinguishable from the Gaussian curve revealing convergence to Gaussian behavior. (b) The moments for  $n=1, 16$ , and 256. These results can be compared with Fig. 8. Note that for the S&P 500 even for time scales  $\Delta t=16$  d (corresponding to  $n=6,240$ ), we still do not observe a good degree of convergence.

Figure 12(a) shows the cumulative distribution of  $G_n^{\text{sh}}(t)$  for increasing values of  $n$ . We find a progressive convergence to Gaussian behavior with increasing  $n$ . This convergence to Gaussian behavior is also clear in the moments of  $G_n^{\text{sh}}(t)$ , which rapidly approach the Gaussian values with increasing  $n$  [Fig. 12(b)]. This rapid convergence confirms that the time dependencies cause the observed scaling behavior.

## VIII. DISCUSSION

We have presented a detailed analysis of the distribution of returns for market indices, for time intervals  $\Delta t$  ranging over roughly four orders of magnitude, from 1 min up to one month ( $\approx 16000$  min). We find that the distribution of returns is consistent with a power-law asymptotic behavior, characterized by an exponent  $\alpha \approx 3$ , well outside the stable Lévy regime  $0 < \alpha < 2$ . For time scales  $\Delta t \gg (\Delta t)_\times$ , where  $(\Delta t)_\times \approx 4$  d, our results are consistent with slow convergence to Gaussian behavior.

We have also demonstrated that the scaling behavior does not hold if we destroy all the time dependencies by shuffling. The breakdown of the scaling behavior of the distribution of returns upon shuffling the time series suggests that the long-range volatility correlations, which persist up to several

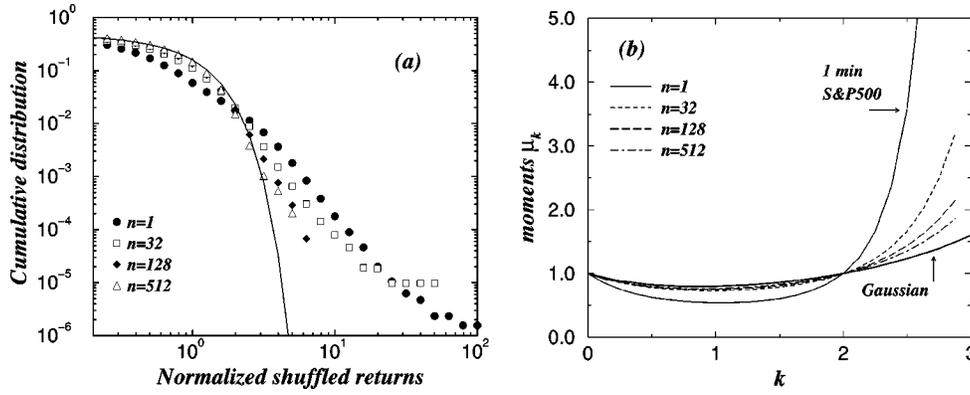


FIG. 12. We randomize the time series of returns for the S&P 500 for  $\Delta t = 1$  min, and create a time series with the same distribution but with independent random variables. We then sum up  $n$  consecutive shuffled returns to create a shuffled  $n$ -min return. (a) Cumulative distributions of the positive tails of the shuffled returns are shown for increasing  $n$ . We find slow convergence to Gaussian behavior on increasing  $n$ . (b) The slow convergence to a Gaussian behavior is shown by the moments. The results in (b) can be compared with Fig. 11(b) if we note that  $n = 512$  corresponds to  $\Delta t \approx 1.5$  d. The data are normalized to have the same second moment.

months [33,53–63], may be one possible reason for the observed scaling behavior.

Recent studies [59] show that the distribution of volatility is consistent with an asymptotic power-law behavior with exponent 3, just as observed for the distribution of returns. This finding suggests that the process of returns may be written as

$$g(t) = \epsilon(t) v(t), \quad (11)$$

where  $g(t)$  denotes the return at time  $t$ ,  $v(t)$  denotes the volatility, and  $\epsilon(t)$  is an i.i.d. random variable independent of  $v(t)$  [14,20,30,66]. Since the asymptotic behavior of the distributions of  $v(t)$  and  $g(t)$  is consistent with power-law behavior,  $\epsilon(t)$  should have an asymptotic behavior with faster decay than either  $g(t)$  or  $v(t)$ . In fact, Eq. (11) is central to all the ARCH models [30,66], with  $\epsilon(t)$  assumed to be Gaussian distributed.

Different ARCH processes assume different recursion relations for  $v(t)$ . In the standard ARCH model,  $v(t) = \alpha + \beta g^2(t-1)$ , leading to a power-law distribution of returns with exponent depending on the parameters  $\alpha$  and  $\beta$ . However, the standard ARCH process predicts volatility correlations that decay exponentially, since  $v(t)$  depends only on the previous event. Thus the ARCH model cannot account for the observed long-range persistence in  $v(t)$ . To try to remedy this, one can require  $v(t)$  to depend not only on the previous value of  $g(t)$  but on a finite number of past events. This generalization is called the GARCH model. Dependence of  $v(t)$  on the finite past leads not to a power-law decay (as is observed empirically), but to volatility correlations that decay exponentially—with larger decay times as the number of events “remembered” is increased.

In order to explain the long-range persistence of the autocorrelation function of the volatility, one must assume that  $v(t)$  depends on all the past rather than a finite number of past events. Such a description [67] would be consistent with the empirical finding of long-range correlations in the volatility, and the observation that the distributions of  $g(t)$  and  $v(t)$  have similar asymptotic forms. Another approach proposed in Ref. [14] is a hierarchical model wherein the volatility correlations decay logarithmically, which is also com-

patible with the power-law decay of volatility correlations with a small exponent. A microscopic model that is consistent with the asymptotic behavior of the distribution of price fluctuations and the long-range correlations in volatility was developed in Refs. [16,17]. If the process of returns were governed by the volatility, as in Eq. (11), then the volatility would seem to be the more fundamental process. In fact, since the volatility is related to the amount of information arriving into the market, the statistical properties of the returns may be “driven” by this information.

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## APPENDIX A: LÉVY STABLE DISTRIBUTIONS

Lévy stable distributions arise from the generalization of the central limit theorem to a wider class of distributions. Consider the partial sum  $P_n \equiv \sum_{i=1}^n x_i$  of independent identically distributed (i.i.d.) random variables  $x_i$ . If the  $x_i$ 's have finite second moments, the central limit theorem holds and  $P_n$  is distributed as a Gaussian in the limit  $n \rightarrow \infty$ .

If the random variables  $x_i$  are characterized by a distribution having asymptotic power-law behavior

$$P(x) \sim x^{-(1+\alpha)}, \quad (A1)$$

where  $\alpha < 2$ , then  $P_n$  will converge to a Lévy stable stochastic process of index  $\alpha$  in the limit  $n \rightarrow \infty$ .

Except for special cases, such as the Cauchy distribution, Lévy stable distributions cannot be expressed in closed form. They are often expressed in terms of their Fourier transforms or characteristic functions, which we denote  $\varphi(q)$ , where  $q$

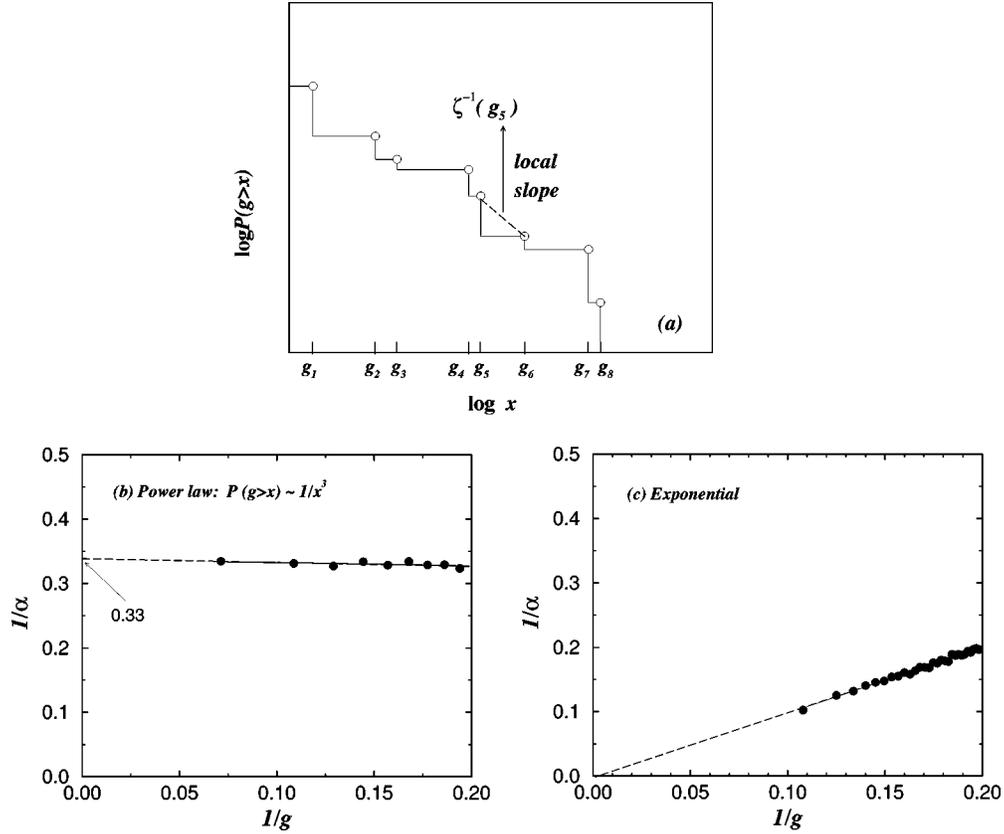


FIG. 13. (a) Schematic representation of the evaluation of the local slope from the cumulative distribution. First, the normalized returns  $g$  are sorted in descending order,  $g_k > g_{k+1}$ . The dotted line indicates the local slope. (b) Hill estimator for a sequence of i.i.d. random variables with asymptotic behavior:  $P(g > x) = (1+x)^{-3}$ . (c) Hill estimator for a sequence of i.i.d. random variables with asymptotic behavior:  $P(g > x) = \exp(-x)$ . Note that the asymptotic estimates  $1/\alpha = 0.33$  and  $1/\alpha = 0$  recover for both cases the correct values of  $\alpha$ ,  $\alpha = 3$  and  $\alpha = \infty$ , respectively.

denotes the Fourier transformed variable. The general form of a characteristic function of a Lévy stable distribution is

$$\ln \varphi(q) = \begin{cases} i\mu q - \gamma |q|^\alpha \left[ 1 + i\beta \frac{q}{|q|} t g \left( \frac{\pi}{2} \alpha \right) \right] & [\alpha \neq 1] \\ i\mu q - \gamma |q| \left[ 1 + i\beta \frac{q}{|q|} \frac{2}{\pi} \ln |q| \right] & [\alpha = 1], \end{cases} \quad (\text{A2})$$

where  $0 < \alpha \leq 2$ ,  $\gamma$  is a positive number,  $\mu$  is the mean, and  $\beta$  is an asymmetry parameter. For symmetric Lévy distributions ( $\beta = 0$ ), one has the functional form

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\gamma |q|^\alpha) e^{-iqx} dq. \quad (\text{A3})$$

For  $\alpha = 1$ , one obtains the Cauchy distribution, and, for the limiting case  $\alpha = 2$ , one obtains the Gaussian distribution.

By construction, Lévy distributions are stable, that is, the sum of two independent random variables  $x_1$  and  $x_2$ , characterized by the same Lévy distribution of index  $\alpha$ , is itself characterized by a Lévy distribution of the same index. The functional form of the distribution is maintained, if we sum up independent, identically distributed Lévy stable random variables.

For Lévy distributions, the asymptotic behavior of  $P(x)$  for  $x \gg 1$  is a power law,

$$P(x) \sim x^{-(1+\alpha)}. \quad (\text{A4})$$

Hence the second moment diverges. Specifically,  $E\{|x|^n\}$  diverges for  $n \geq \alpha$  when  $\alpha < 2$ . In particular, all Lévy stable processes with  $\alpha < 2$  have *infinite* variance. Thus non-Gaussian stable stochastic processes do not have a characteristic scale. Although well defined mathematically, these distributions are difficult to use and raise fundamental problems when applied to real systems where the second moment is often related to the properties of the system. In finance, an infinite variance would make risk estimation and derivative pricing impossible.

## APPENDIX B: HILL ESTIMATOR (“LOCAL SLOPES”)

A common problem when studying a distribution that decays as a power law is how to obtain an accurate estimate of the exponent characterizing the asymptotic behavior. Here we review the methods of Hill [68]. The basic idea is to calculate the *inverse* of the local logarithmic slope  $\zeta$  of the cumulative distribution  $P(g > x)$ ,

$$\zeta \equiv - \left( \frac{d \ln P}{d \ln x} \right)^{-1}. \quad (\text{B1})$$

We then estimate the inverse asymptotic slope  $1/\alpha$  by extrapolating  $\zeta$  as  $1/x \rightarrow 0$ . We start with the normalized returns  $g$  and proceed in the following steps:

*Step I:* We sort the normalized returns  $g$  in descending order. The sorted returns are denoted  $g_k$ ,  $k=1, \dots, N$ , where  $g_k > g_{k+1}$  and  $N$  is the total number of events.

*Step II:* The cumulative distribution is then expressed in terms of the sorted returns as

$$P(g > g_k) = \frac{k}{N}. \quad (\text{B2})$$

Figure 13 is a schematic of the cumulative distribution thus obtained. The *inverse* local slopes  $\zeta(g)$  can be written as

$$\zeta(g_k) = -\frac{\ln(g_{k+1}/g_k)}{\ln(P(g_{k+1})/P(g_k))}. \quad (\text{B3})$$

Using Eq. (B2), the above expression can be well approximated for large  $k$  as

$$\zeta(g_k) \approx k(\ln(g_{k+1}) - \ln(g_k)), \quad (\text{B4})$$

yielding estimates of the local inverse slopes.

*Step III:* We obtain the inverse local slopes through Eq. (B4). We can then compute an average of the inverse slopes over  $m$  points,

$$\langle \zeta \rangle \equiv \frac{1}{m} \sum_{k=1}^m \zeta(g_k), \quad (\text{B5})$$

where the choice of the averaging window length  $m$  varies depending on the number of events  $N$  available.

*Step IV:* We plot the locally averaged inverse slopes  $\langle \zeta \rangle$  obtained in step III as a function of the inverse normalized returns  $1/g$  (see, e.g., Fig. 5). We can then define two methods of estimating  $\alpha$ . In the first method, we extrapolate  $\zeta$  as a function of  $1/g$  to 0, similarly to the method of successive slopes [69]; this procedure yields the inverse asymptotic slope  $1/\alpha$ . In the second method, we average over all events for  $1/g$  smaller than a given threshold [68], with the average yielding the inverse slope  $1/\alpha$ .

To test the Hill estimator, we analyze two surrogate data sets with known asymptotic behavior: (a) an independent random variable with  $P(g > x) = (1+x)^{-3}$ , and (b) an independent random variable with  $P(g > x) = \exp(-x)$ . As shown in Figs. 13(b) and 13(c), the method yields the correct results  $\alpha = 3$  and  $\alpha = \infty$ , respectively.

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