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Turbulent Cascades in Foreign Exchange Markets

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Abstract

The availability of high-frequency data for financial markets has made it possible to study market dynamics on timescales of less than a day [1]. For foreign exchange (FX) rates Müller et al. [2] have shown that there is a net information flow from long to short time scales: the behaviour of long-term traders (who watch the market only from time to time) influences the behaviour of short-term traders (who watch the market continuously). Motivated by this hierarchical feature, we have studied FX market dynamics in more detail, and report here an analogy between FX market dynamics and hydrodynamic turbulence [3–8]. Specifically, the relationship between the probability density of FX price changes (Δx) and the time delay (Δt) (Fig. 1a) is much the same as the relationship between the probability density of the velocity differences (Δv) of two points in a turbulent flow and their spatial distance Δr (Fig. 1b). Guided by this similarity we claim that there is an information cascade in FX market dynamics that corresponds to the energy cascade in hydrodynamic turbulence. On the basis of this analogy we can now rationalize the statistics of FX price differences at different time delays, which is important for, for example, option pricing. The analogy also provides a conceptual framework for understanding the short-term dynamics of speculative markets.

A flow in space from large to small scales, similar to the information flow in time, is the energy cascade which represents the main characteristics of fully-developed homogeneous isotropic turbulence in three spatial dimensions [24]. It provides a mechanism for dissipating large amounts of energy in a viscous fluid. Energy is pumped into the system at large scales of the order of, say, meters (by a moving car or a flying airplane) or kilometers (by meteorological events), transferred to smaller scales through a hierarchy of eddies of decreasing sizes and dissipated at the smallest scale of the order of millimeters in the above examples. This cascade of kinetic energy extending over several orders of magnitude generates a scaling behaviour of the eddies and manifests itself in a scaling of the moments $\langle(\Delta v)^n\rangle$ of Δv as $(\Delta r)^{\zeta_n}$ [5,6]. Here the angle brackets $\langle\ \rangle$ denote the mean value of the enclosed quantity and Δv is the difference of the velocity component in the direction of the spatial separation of length Δr. Under the assumption that the eddies of each size are space-filling and that the downward energy flow is homogeneous, $\zeta_n = n/3$ [8]. The probability densities $P_{\Delta r}(\Delta v)$ are then scale invariant. This means that if the velocitiy differences are normalized by their respective standard deviation, the resulting standardized probability densities do not depend on Δr. But for $n > 3$, experimentally determined values of ζ_n follow a concave curve definitely below the ($\zeta_n = n/3$)-line (Fig. 2b). The dependence of the standardized probability density on Δr also provides evidence that eddies of a given size are not space-filling but rather fluctuating in space and time in a typical intermittent way, see (Fig.1b).

Our analyses of FX markets are based on a data set provided by Olsen & Associates containing all worldwide 1 472 241 bid–ask quotes for US Dollar-German mark exchange rates which have emanated from the interbank Reuters network from 1 October 1992 until 30 September 1993. From these data we have determined the probability densities of price changes $P_{\Delta t}(\Delta x)$ with time delays varying from five minutes up to approximately two days, which are displayed in Fig. 1a. In comparison Fig. 1b shows the analogous turbulent probability densities $P_{\Delta r}(\Delta v)$, which exhibit the same characteristic features. Using the probability density $P_{\Delta t}(\Delta x)$, the information acquired by observing the market after a time Δt can be quantified as $I(\Delta t) = -\int P_{\Delta t}(\Delta x)\log P_{\Delta t}(\Delta x)\, d(\Delta x)$. It turns out that the dependence of this information on Δt is directly related to the scaling of the variance of Δx with Δt. In turbulence, on the other hand, the variance of the velocity differences at a distance Δr is proportional to the mean energy which is contained in an eddy of size Δr. This further supports the proposed analogy between energy and information.

Given the analogy between turbulence and FX market dynamics, we expect the moments of FX price changes to scale with the time delay as $\langle(\Delta x)^n\rangle \propto (\Delta t)^{\xi_n}$. Scaling has already been reported for the mean absolute values in FX returns in ref. [9] and by Evertsz in ref. [1], and for the 2nd moments of the variations of the Standard & Poor's 500 economic index in ref. [10]. For FX market data, it has also been observed that the kurtosis, which is the ratio of the fourth moment and the squared variance, decreases with increasing time delay, that is, the tails of the probability density lose weight [9,11]. Figure 2a shows the higher-order moments of the FX price changes. They scale for time delays Δt varying from about 5 minutes up to several hours. As in turbulence, the scaling exponents ξ_n depend on the order of the moments in a nonlinear way. Moreover, the ξ_n of the FX data are close to the ζ_n of turbulent data (Fig. 2b). This is also in agreement with the observation that the shapes of the turbulent and FX probability densities, $P_{\Delta r}(\Delta v)$ and $P_{\Delta t}(\Delta x)$, depend on their respective scale parameters in a similar way: both show a decrease of kurtosis with increasing scale parameter [12] (Figs. 1a and 1b).

Even though the probability densities $P_{\Delta t}(\Delta x)$ and $P_{\Delta r}(\Delta v)$ are in principle determined by their moments, this is of little help in practice because errors drastically increase when calculating higher-order moments from observed data. In turbulence, for different experimental situations, the change in shape of $P_{\Delta r}(\Delta v)$ has been successfully parameterized using a method motivated by the energy cascade [7,13,14]. The standardized probability density is approximated by a superposition of Gaussian densities with log-normally distributed variances. The variance λ^2 of this log–normal distribution is a measurable form parameter containing information on the energy cascade [15]. It is the only parameter that must be adjusted to the data. Applying this method to the FX market data yields a surprisingly good fit (Fig. 1a). Although the data in the center dominate the fit, the agreement is reasonable also as regards the tails of the probability densities as long as the time delay is shorter than about two days. This is in contrast to fits with Levy distributions which significantly deviate in the tails [10]. The parameter λ^2, which measures the spread of the variances of the superimposed Gaussians (that is, the variance of the log-normal distribution), decreases linearly with $\log \Delta t$ (Fig. 3). This result further confirms the similarity of the statistical behaviour of FX markets with the classical picture of turbulence as given by Kolmogorov [5–7]. In particular, it verifies the scaling of the moments, $\langle(\Delta x)^n\rangle \propto (\Delta t)^{\xi_n}$, in an independent way.

An important aspect of turbulent flows is their intermittent behaviour, that is, the typical occurence of laminar periods which are interrupted by turbulent bursts. In FX markets this corresponds to clusters of high and low volatility [12], which give rise to relatively high values of the probability densities $P_{\Delta t}(\Delta x)$ both in the center and the tails. This particular aspect is well reproduced for time delays smaller than two days by the proposed log-normal superposion of Gaussians. However, long interruptions of the trading process (particularly during weekends) affect the probability densities and lead to deviations from the expected shape. Hence, we conclude that the range which is governed by the information cascade has an upper limit. We note that in turbulence also there is an upper length scale, where the energy cascade sets in and beyond which scaling fails.

Different types of models have been proposed to describe the statistical characteristics of price differences of financial indices as, for example, the behaviour of the volatility. Prominent approaches are models using mixtures of distributions [16–18] and ARCH/GARCH-type models [19,20] (see, for example, the overview in ref. [21]). But these studies do not address the scaling behaviour of the probability distributions of FX price changes.

It is unlikely that there is a set of a few partial differential equations (like the Navier Stokes equations in hydrodynamics) which might serve as a model of FX market dynamics. More striking is the similarity of the two phenomena, which is accounted for by the existence of a cascade in both cases. This has prompted us to introduce concepts of turbulence in the description of FX markets, in particular that of an information cascade. Table I summarizes the corresponding notions. How far this analogy goes still has to be shown. In any case, we have reason to believe that the qualitative picture of turbulence that has developed during the past 70 years will help our understanding of the apparently remote field of financial markets.

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FIGURES

FIG. 1. (a) Data points: Standardized probability density $P_{\Delta t}(\Delta x)$ of price changes $\Delta x = x(t) - x(t+\Delta t)$ for time delays $\Delta t = 640s, 5120s, 40960s, 163840s$ (from top to bottom). The middle prices $x(t) = (x_{bid}(t) + x_{ask}(t))/2$ have been used (data are provided by *Olsen & Associates* (see text)). The probability density has been calculated in a similar way as in ref. [10]. Full lines: results of (least-squares) fits carried out according to ref. [7]; $\lambda^2 = 0.25, 0.23, 0.13, 0.06$ (from top to bottom). For better visibility, the curves have been vertically shifted with respect to each other. Note the systematic change in shape of the densities.
(b) Data points: standardized probability density $P_{\Delta r}$ of velocity differences Δv for a turbulent flow with $\Delta r = 3.3\eta, 18.5\eta, 138\eta, 325\eta$ (data taken from ref. [13], $R_\lambda = 598$). Here η is the Kolmogorov scale, where viscous dissipation occurs. Full lines: results of (least-squares) fits carried out according to ref. [7]; $\lambda^2 = 0.19$, 0.10, 0.04, 0.01.

FIG. 2. (a) Scaling of moments $\langle(\Delta x)^n\rangle$ with $n = 2$, 4, and 6.
(b) n-dependence of the scaling factors ξ_n and ζ_n for the nth moments of the probability densities of FX price changes (squares) and hydrodynamic velocity differences taken from ref. [22] (crosses) and ref. [23] (triangles). Note the same qualitative deviation of all curves from a straight line.

FIG. 3. Dependence of the form parameter λ^2 on Δt (in seconds).

TABLES

TABLE I. Correspondence between fully-developed three-dimensional turbulence and FX markets

hydrodynamic turbulence	FX markets
energy	information
spatial distance	time delay
laminar periods interrupted by turbulent bursts (intermittency)	clusters of low and high volatility
energy cascade in space hierarchy	information cascade in time hierarchy
$\langle(\Delta v)^n\rangle \propto (\Delta r)^{\zeta_n}$	$\langle(\Delta x)^n\rangle \propto (\Delta t)^{\xi_n}$