arXiv:cond-mat/9708012 v1 1 Aug 1997

Causal cascade in the stock market from the "infrared" to the "ultraviolet".

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Modelling accurately financial price variations is an essential step underlying portfolio allocation optimization, derivative pricing and hedging, fund management and trading. The observed complex price fluctuations guide and constraint our theoretical understanding of agent interactions and of the organization of the market. The gaussian paradigm of independent normally distributed price increments [1, 2] has long been known to be incorrect with many attempts to improve it. Econometric nonlinear autoregressive models with conditional heteroskedasticity[3] (ARCH) and their generalizations [4] capture only imperfectly the volatility correlations and the fat tails of the probability distribution function (pdf) of price variations. Moreover, as far as changes in time scales are concerned, the so-called "aggregation" properties of these models are not easy to control. More recently, the leptokurticity of the full pdf was described by a truncated "additive" Lévy flight model[5, 6] (TLF). Alternatively, Ghashghaie *et al.*[7] proposed an analogy between price dynamics and hydrodynamic turbulence.

In this letter, we use wavelets to decompose the volatility of intraday (S&P500) return data across scales. We show that when investigating two-points correlation functions of the volatility logarithms across different time scales, one reveals the existence of a causal information cascade

from large scales (i.e. small frequencies, hence to vocable "infrared") to fine scales ("ultraviolet"). We quantify and visualize the information flux across scales. We provide a possible interpretation of our findings in terms of market dynamics.

The controversial [6, 8] analogy developed by Ghashghaie *et al.*[7] implicitly assumes that price fluctuations can be described by a *multiplicative cascade* along which, the return at a given scale $a < T$, is given by:

$$r_a(t) \equiv \ln P(t+a) - \ln P(t) = \sigma_a(t) u(t) \; , \tag{1}$$

where $u(t)$ is some scale independent random variable, T is some coarse "integral" time scale and $\sigma_a(t)$ is a positive quantity that can be multiplicatively decomposed, for each decreasing sequence of scales $\{a_i\}_{i=0,..,n}$ with $a_0 = T$ and $a_n = a$, as[9, 10]

$$\sigma_a = \prod_{i=0}^{n-1} W_{a_{i+1},a_i} \sigma_T \; . \tag{2}$$

In turbulence, the field σ is related to the energy while in finance σ is called the volatility. Recall that the volatility has fundamental importance in finance since it provides a measure of the amplitude of price fluctuations, hence of the market risk. Using $\omega_a(t) \equiv \ln \sigma_a(t)$ as a natural variable, if one supposes that W_{a_{i+1},a_i} depends only on the scale ratio a_i/a_{i+1}, one can easily show, by choosing the a_i as a geometric series Ts^n $(s < 1)$, that eq. (2) implies that the pdf of ω at scale a can be written as[9, 10]

$$p_a(\omega) = (G_s^{\otimes n} \otimes p_T)(\omega) \; , \tag{3}$$

where $\otimes$ means the convolution product, G_s is the pdf of $\ln W_{sa,a}$ and p_T is the pdf of ω_T. The above equation is the exact reformulation (in log variables) of the paradigm that Ghashghaie *et al.* [7] used to fit foreign exchange (FX) rate data at different scales. In this formalism, G can be proven to be the pdf of an infinitely divisible random variable [10] (hence σ is called "log-infinitely divisible"). In ref. [7], G is assumed to be Normal (the cascade is called "log-normal") of variance $-\lambda^2 \ln s$.

First, let us comment on the criticisms raised by Mantegna and Stanley [8]. Note that eq. (3) does not determine the shape of the pdf of the returns $r_a(t)$ *at a given scale* but specifies how this pdf changes *across scales*. For a fixed scale, the precise form for the pdf depends on both p_T and on the law of the variable $u(t)$ (which determines notably the sign of $r_a(t)$). Therefore, nothing prevents the pdf of $r_a(t)$ to having fat tails at small scales as observed in financial time series [7]. A cascade model actually accounts for the distribution of the volatility of returns across scales and not for the precise fluctuations of $r_a(t)$. The behavior of the autocorrelation function $\overline{r_a(t)r_a(t+\tau)}$ $(\tau > a)$ indeed depends on both the cascade variables and $u(t)$. For example, if $u(t)$ is a white noise, there will be no correlation between the returns while their absolute values (or the associated volatilities) are strongly correlated (see below). This is why the shape of the power spectrum of financial time series cannot be invoked as an argument against a cascade model. Moreover, as far as scaling properties of price fluctuations are concerned, it is easy to deduce from eq. (3) that, if $H \ln s$ is the mean of G_s and $-\lambda^2 \ln s$ its variance, then the the maximum of the pdf of $\sigma_a(t)$ varies as $a^{H-\lambda^2/2}$ (H plays the same role as the Lévy index in TLF models with $H = 1/\mu$) while its standard deviation behaves as $a^{H-\lambda^2}$; these features are observed in both turbulence [9] ($H \simeq 0.33$ and $\lambda^2 \simeq 0.03$) and finance [7] ($H \simeq 0.6$ and $\lambda^2 \simeq 0.015$). Therefore, as advocated in ref. [7], eq. (3) accounts reasonably well for one-point statistical properties of financial times series. However, because of the relatively small statistics available in finance, it is very difficult to demonstrate that eq. (3) is more pertinent to fit the data than a "truncated Lévy" distribution [5, 6, 8].

At this point, let us emphasize that eq. (2) imposes much more constraints on the statistics (it is indeed a model !) than eq. (3) that only refers to one point statistics. The main difference between the *multiplicative* cascade model and the truncated Lévy *additive* model is that the former predicts strong correlations in the volatility while the latter assumes no correlation. It is then tempting to compute the correlations of the log-volatility ω_a at different time scales a. For that purpose, we use a natural tool to perform time-scale analysis, the *wavelet transform* (WT). Wavelet analysis has been introduced as a way to decompose signals in both time and scales [11]. The WT of $f(t) = \ln P(t)$ is defined as:

$$T_\psi[f](t,a) \equiv \frac{1}{a}\int_{-\infty}^{+\infty} f(y)\psi\left(\frac{y-t}{a}\right)dy, \tag{4}$$

where t is the time parameter, a (>0) the scale parameter and ψ the analyzing wavelet. Note that for $\psi(t) = \delta(t-1) - \delta(t)$, $T_\psi[f](t,a)$ is nothing but the return $r_a(t)$. However, in general, ψ is choosen to be well localized in both time and frequency, so that the scale a can be interpreted as an inverse frequency. Moreover, if ψ has at least two vanishing moments and χ is a bump function with $||\chi||_1 = 1$, then, the *local* volatility at scale a and time t can be defined as [12] $\sigma_a^2(t) \equiv a^{-3} \int \chi((b-t)/a) |T_\psi(b,a)|^2 db$. Actually, thanks to the time-scale properties of the wavelet decomposition [11], when summing $\sigma_a^2(t)$ over time and scale, one recovers the total square derivative of f: $\Sigma = \int \int \sigma_a^2(t) dt da = \int |df/dt|^2 dt$.

In Fig. 1 are shown 3 time series for which we study the increment time correlations. Fig. 1(a) represents the logarithm of the $S\&P500$ index. The corresponding "volatility walk", $v_a(t) = \sum_{i=0}^{t} \omega_a(i)$ is represented in Fig. 1(b). Fig. 1(c) is the same as Fig. 1(b) but after having randomly shuffled the increments $\ln P(i+1) - \ln P(i)$ of the signal in Fig. 1(a). Fig. 1(b) clearly demonstrates the existence of important long-range positive temporal correlations in the volatilities of $S\&P500$ returns. Moreover, the statistics of $\omega_a(t)$ are found to be nearly gaussian. However, the volatility walk for the "shuffled S&P500" looks very much like a Brownian motion with uncorrelated increments. This observation is sufficient to discard any additive (like TLF) model which intrinsically fails to account for the strong correlations observed in $\omega_a(t)$. The correlation function $C_1^r(\Delta t) = \overline{r_1(t) r_1(t+\Delta t)} - \overline{r_1(t)}^2$ shown in Fig. 1(a'), confirms the well-known fact that there are no correlations between the returns (except at a very small time lag as illustrated in the inset). However, the difference is striking in Fig. 1(b') where the correlation function of the volatility walk $C_a^\omega(\Delta t) = \overline{\omega_a(t)\omega_a(t+\Delta t)} - \overline{\omega_a(t)}^2$ remains as large as 5% up to time lags corresponding to about two months. In contrast, the correlation function associated to the shuffled time series in Fig. 1(c') is within the noise level.

From the modelling of fully developed turbulent flows and fragmentation processes, random multiplicative cascade models are well known to generate long-range correlations [13, 14, 15]. We now explore whether this concept could be useful for understanding the observed long-range correlations of the volatility (and not of the price increments, which makes turbulence and financial markets drastically different).

To fix ideas, let us consider a specific realization of a process satisfying eq. (2). Consider the largest time scale T of the problem. We then assume that the volatility at time scale T influences the volatility of the two subperiods of length $\frac{T}{2}$ by random factors equal respectively to W_0 and W_1. In turn, each volatility over $\frac{T}{2}$ influences the two subperiods of length $\frac{T}{4}$ by random factors W_{00} and W_{10} for the first sub-period and W_{01} and W_{11} for the second one. The cascade process is assumed to continue along the time scales until the shortest tick time scale (see ref. [10] for rigourous definitions and properties). The simplest assumption is that the factors W are i.i.d. variables with log-normal distribution of mean $-H \ln 2$ and variance $\lambda^2 \ln 2$. It is then easy to show that the correlation function averaged over a period of length T, $C_a^\omega(\Delta t) = T^{-1} \int_0^T \left(\langle \omega_a(t) \omega_a(t+\Delta t) \rangle - \langle \omega_a(t) \rangle^2 \right) dt$, can be written as

$$C_a^\omega(\Delta t) = \lambda^2 (1 - \log_2 \frac{\Delta t}{T} - 2\frac{\Delta t}{T}) \ , \tag{5}$$

for $a \leq \Delta t \leq T$ ($\langle . \rangle$ means mathematical expectation). Here, our goal is to show that the basic ingredients of this simple cascade model are sufficient to rationalize most of the features observed on the volatility correlations at different scales (note that one could improve this description by taking into account mutual influences of volatilities at a given scale and the possible "inverse cascade" influence of fine scales on larger ones). For $\lambda^2 \simeq 0.015$ obtained independently from the fit of the pdf's [7], eq. (5) provides a very good fit of the data (Fig 1(b')) for the slow decay of the correlation function with only one adjustable parameter $T \simeq 3$ months. Let us note that $C_a^\omega(\Delta t)$ can be equally well fitted by a power law $\Delta t^{-\alpha}$ with $\alpha \approx 0.2$. In view of the small value of α, this is undistinguishable from a logarithmic decay. Moreover, eq. (5) predicts that the correlation function $C_a^\omega(\Delta t)$ should not depend of the scale a provided $\Delta t > a$. In Fig. 2, $C_a^\omega(\Delta t)$ are plotted versus $\ln(\Delta t)$ for various scales a corresponding to 30, 120 and 480 min. As expected, all the data collapse on a single curve which is nearly linear up to some integral time of the order of 3 months.

Let us point out that volatility at large time intervals that cascades to smaller scales cannot do so instantaneously. From causality properties of financial signals, the "infrared" towards "ultraviolet" cascade must manifest itself in a time asymmetry of the cross-correlation coefficients $C_{a_1,a_2}^\omega(\Delta t) \equiv \mathrm{var}(\omega_{a_1})^{-1} \mathrm{var}(\omega_{a_2})^{-1} (\overline{\omega_{a_1}(t) \omega_{a_2}(t+\Delta t)} - \overline{\omega_{a_1}(t)} \ \overline{\omega_{a_2}(t)})$; in particular, one expects that $C^{\omega}_{a_1,a_2}(\Delta t) > C^{\omega}_{a_1,a_2}(-\Delta t)$ if $a_1 > a_2$ and $\Delta t > 0$. From the near-Gaussian properties of $\omega_a(t)$, the mean mutual information of the variables $\omega_a(t+\Delta t)$ and $\omega_{a+\Delta a}(t)$ reads:

$$I_a(\Delta t, \Delta a) = -0.5 \log_2 \left(1 - (C^{\omega}_{a,a+\Delta a}(\Delta t))^2\right) \ . \tag{6}$$

Since the process is causal, this quantity can be interpreted as the information contained in $\omega_{a+\Delta a}(t)$ that propagates to $\omega_a(t+\Delta t)$. In Fig. 3, we have computed $I_a(\Delta t, \Delta a)$ for the S&P500 index (top) and its randomly shuffled version (bottom). One can see on the bottom picture that there is no well defined structure that emerges from the noisy background. Except in a small domain at small scales around $\Delta t = 0$, the mutual information is in the noise level as expected for uncorrelated variables. In contrast, two features are clearly visible on the top representation. First, the mutual information at different scales is mostly important for equal times. This is not so surprising since there are strong localized structures in the signal that are "coherent" over a wide range of scales. The extraordinary new fact is the appearance of a non symmetric propagation cone of information showing that the volatility a large scales influences causally (in the future) the volatility at shorter scales. Although one can also detect some information that propagates from past fine to future coarse scales, it is clear that this phenomenon is weaker than past coarse/future fine flux (the fact that the former one exists anyway suggests that a more realistic cascading process should include the causal influence of short time scales on larger ones). Figure 3 is thus a clear demonstration of the pertinence of the notion of a cascade in market dynamics. Similar features have been found on Foreign Exchange rates.

There are several mechanisms that can be invoked to rationalize our observations, such as the heterogeneity of traders and their different time horizon [16] leading to an "information" cascade from large time scales to short time scales, the lag between stock market fluctuations and long-run movements in dividends [17], the effect of the regular release (monthly, quarterly) of major economic indicators which cascades to fine time scale. Correlations of the volatility have been known for a while and have been partially modelled by mixtures of distributions [18], ARCH/GARCH models [3] and their extensions [4]. However, as pointed out in the introduction, because they are constructed to fit the fluctuations at a given time interval, these models are not adapted to account for the above described multi-scale properties of financial time series. We have performed the same correlation analysis for simulated GARCH(1,1) processes and obtained structureless pictures similar to the one corresponding to the shuffled S&P500 in Fig. 3(b). More recently, Muller *et al.* [16] have proposed the HARCH model in which the variance at time t is a function of the realized variances at different scales. By construction, this model captures the lagged correlation of the volatility from the large to the small time scales. However, it does not contain the notion of cascade and involves only a few time scales. Moreover, it suffers from the same defficiencies as ARCH-type models concerning the difficulties to control and interpret parameters at different scales.

Putting together the evidence provided by the logarithmic decay of the volatility correlations and the volatility cascade from the infrared to the ultraviolet, we have revisited the analogy with turbulence, albeit on the *volatility* and not on the price variations. Another very promising prospect consists in building ARCH-type processes on orthogonal wavelets basis. This work is in current progress. The present understanding with such models will allow us to calculate improved risk prices such as options, for instance using the functional formalism of ref. [19] well-adapted to deal with pdf's of the form (3).

Acknowledgments. We acknowledge useful discussions with E. Bacry and U. Frisch.

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Figure Captions

Figure 1: (a) Time evolution of $\ln P(t)$, where $P(t)$ is the S&P500 index, sampled with a time resolution $\delta t = 5$ min in the period October 1991-February 1995. The data have been preprocessed in order to remove "parasitic" daily oscillatory effects. (b) The corresponding "volatility walk", $v_a(t) = \sum_{i=0}^{t} \omega_a(i)$, as computed with a compactly supported spline wavelet[11] for $a = 4$ ($\simeq$ 20 min). (c) $v_a(t)$ computed after having randomly shuffled the increments of the signal in (a). (a') The 5 min return correlation function $C_1^r(\Delta t)$ versus Δt from 0 to 20 min. (b') The correlation function $C_a^\omega(\Delta t)$ of the log-volatility of the S&P500 at scale $a = 4$ ($\simeq$ 20 min); the solid line corresponds to a fit of the data using eq. (5) with $\lambda^2 = 0.015$ and $T \simeq 3$ months. (c') same as in (b') but for the randomly shuffled S&P500 signal. In (a'-c') the dashed lines delimit the 95% confidence interval.

Figure 2: The correlation function $C_a^\omega(\Delta t)$ of the log-volatility of the S&P500 index is plotted versus $\ln \Delta t$ for various scales a corresponding to 30 ($\circ$), 120 ($\times$) and 480 ($\triangle$) minutes. All the data collapse on a same curve which is almost linear up to an integral time scale $T \simeq 3$ months ($\ln T = 8.6$). According to eq. (5), from the slope of this straight line, one gets an estimate of the parameter $\lambda^2 \simeq 0.015$.

Figure 3: The mutual information $I_a(\Delta t, \Delta a)$ (eq. (6)) of the variables $\omega_a(t + \Delta t)$ and $\omega_{a+\Delta a}(t)$ is represented in the $(\Delta t, \Delta a)$ half-plane (5 min units); the time lag Δt spans the interval $[-2048, 2048]$ while the scale lag Δa ranges from $\Delta a = 0$ (top) to 1024 (bottom). The amplitude of $I_a(\Delta t, \Delta a)$ is coded from black for zero values to red for maximum positive values ("heat" code), independently at each scale lag Δa. (a) S&P500 index; (b) its randomly shuffled increment version. Note that, for middle scale lag values, the maxima (red spots) of the mutual information in (a) are 2 order of magnitude larger than the corresponding maxima in (b).

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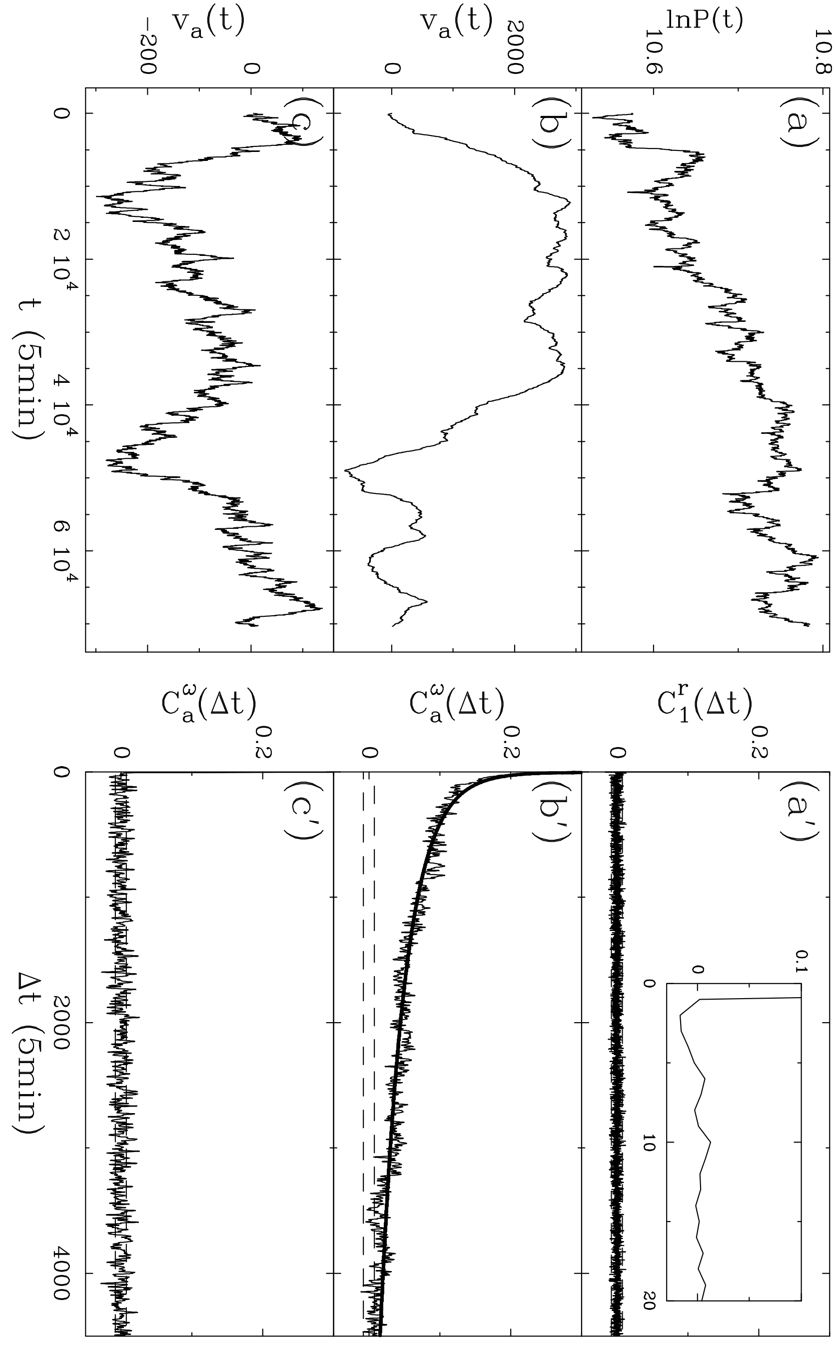

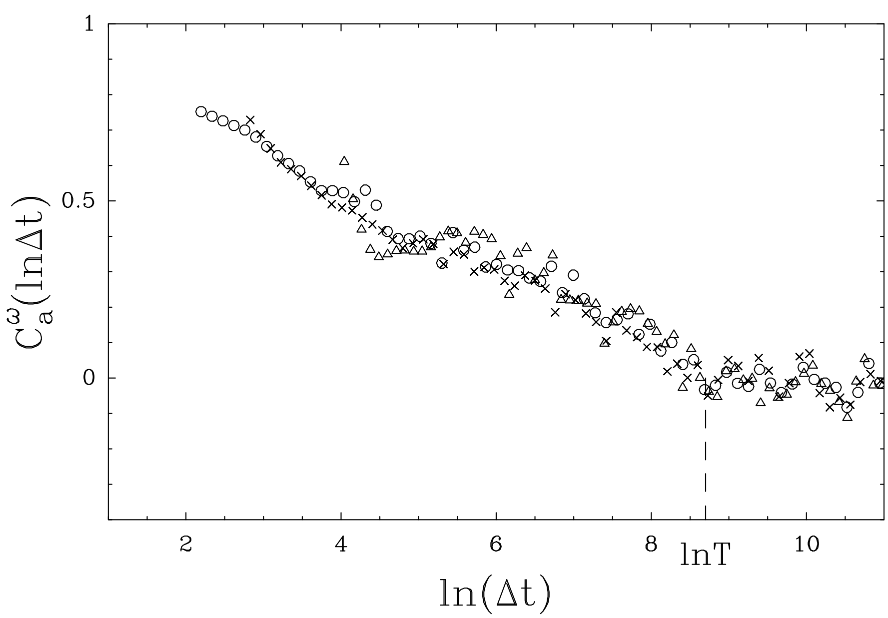

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