Bispectral-Based Tests for the Detection of Gaussianity and Linearity in Time Series

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Statistical techniques have been developed that use estimated bispectrum values to test whether a sample of a time series is consistent with the hypothesis that the observations are generated by a linear process. The magnitude of the test statistics indicates the amount of divergence between the observations and the linear model hypothesis. It is important to investigate such a divergence, since the usual linear model coefficients can be shown to be biased in the face of nonlinear time series structure. The tests presented here can thus be considered diagnostic as well as confirmatory. These tests are applied to a variety of real series previously modeled with linear models. The results indicate nonlinear models may yield better results, because many of the series analyzed appear to have considerable nonlinear lagged interactions.

KEY WORDS: Polyspectra; Statistical test of models; Time series structure.

1. INTRODUCTION

The application of linear time series analysis to real data has increased tremendously in the past 15 years. Applied time series analysis is now a standard feature of undergraduate and graduate curricula in the physical sciences, life sciences, social sciences, engineering, and business education. This phenomenon is a product of three related events: (a) a realization that many scientific problems are amenable to time series analysis, (b) the prodigious research effort directed at linear time series analysis, and (c) the rapid growth in the number of time-based, machine-readable data sets and the availability of linear time series methods on standard statistical software packages.

Nevertheless, it must be emphasized that there is no particular reason why empirical time series should conform to linear time series models, or even be well approximated by a linear model. If the correct model for the time series is nonlinear, then the coefficient fit when using a linear model could be biased, with the degree of bias depending on the extent of nonlinearity.

For example, consider the special case of the quadratic model.

$$x(t) = \sum_{n\geq 0} h(n)\varepsilon(t-n) + \sum_{m\geq 1}\sum_{n\geq 0} a(m,n)\varepsilon(t-n)\varepsilon(t-n-m),$$

where the $\varepsilon(t)$ are unobserved iid zero mean random variates whose cumulants exist. Assume that $\{h(t)\}$ and $\{a(m, n)\}$ are square summable sequences so that the spectrum of $x(t)$ can be expressed in terms of the absolute values of the linear and quadratic transfer functions. Setting the variance of the innovations $\sigma_\varepsilon^2 = 1$ for simplicity, the rth covariance term of the $x(t)$ is $h(r) + \sum_{m\geq 1}\sum_{n\geq 0} a(m, n) \times a(m, n + r)$. This latter expression shows the confounding between the linear impulse-response terms $h(r)$ and the quadratic terms in the covariance function of the observed $x(t)$. Conventional linear model analysis based on sample autocorrelations yields misleading results for such quadratic models.

For diagnostic purposes as well as normative purposes, it is important to discern the extent of nonlinearity in the series and determine whether or not this nonlinearity is significant. If the nonlinearity is significant the process of nonlinear time series modeling must be addressed. Tong (1983), Subba Rao and Gabr (1984), Maravall (1983), and Petruccelli and Davies (1986) showed that when nonlinearity is present, prediction can be improved by using nonlinear models. Because of the broad expanse of nonlinear models (everything not linear), the process of nonlinear time series modeling is advancing on several fronts. For example, Subba Rao and Gabr (1984) developed and fit bilinear models to nonlinear time series, and Priestley (1981) discussed other nonlinear models.

Given the nature of confounding linear and quadratic coefficients in the estimation of time series models, it is important to test for significant nonlinearity in the observed time series. Thus the purpose of this article is not to develop new nonlinear models, but to present statistical techniques for determining which time series are actually linear processes and which time series are not amenable to linear time series modeling. We then apply these time series tests to a variety of real time series previously modeled with linear time series methods.

The stationary autoregressive (AR) and the stationary autoregressive moving average (ARMA) models, which are widely used in time series analysis, are finite-order representations of the general causal linear model. Even if the data appear to have a trend during the observation period, linear-filtering techniques such as differencing (Box and Jenkins 1970, sec. 4.1) and trend regression (Priestley 1981, sec. 7.7) can often be used to transform the data into a seemingly stationary sequence.

The unobserved innovation (input) processes in AR and ARMA models are usually assumed to be white noise;

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that is, the innovations $\varepsilon(t)$ are homoscedastic serially uncorrelated random variables (with zero mean). The statistical properties of the parameter estimators for these models are much simpler if the innovations are *purely random*, that is, the $\varepsilon(t)$ are independent random variables. In many applications the $\varepsilon(t)$ are assumed to be jointly Gaussian, so there is no difference between the white-noise assumption and the purely random one. When the innovations are non-Gaussian, the higher-order dependence among the innovations affects parameter estimators of models estimated from a sample of the $x(t)$'s.

Finite-parameter AR and ARMA models can be represented as a one-sided moving average of purely random noise. Priestley (1981, sec. 3.5.7) called this representation the general linear time series model. The observation of the process $\{X(t)\}$ at the time t can be expressed as the one-sided moving average

$$X(t) = \sum_{s=0}^{\infty} h(s)\varepsilon(t-s), \tag{1.1}$$

where $\{\varepsilon(t)\}$ is a purely random noise process.

Standard analysis of stationary time series data proceeds by estimating the parameters for a relatively low-order linear model using the sample autocovariance (or autocorrelation) function. Some investigators also estimate the spectrum from the data to aid in model identification. The assumption that the unobserved input process $\{\varepsilon(t)\}$ is white is sufficient for identifying the linear structure, because such a task can be done from sample autocovariances.

On the other hand, if the observed process is the output of a nonlinear operation on an input process, then the sample autocovariances are insufficient for identifying the structure of the nonlinear filter. For example, suppose that

$$\begin{aligned} X(t) = {} & 17\varepsilon(t)\varepsilon(t-16) + 17\varepsilon(t)\varepsilon(t-23) \\ & + 17\varepsilon(t)\varepsilon(t-27) + 17\varepsilon(t)\varepsilon(t-30) \\ & + 17\varepsilon(t)\varepsilon(t-36) + 17\varepsilon(t)\varepsilon(t-39) \\ & + 31\varepsilon(t)\varepsilon(t-95) + 31\varepsilon(t-9)\varepsilon(t-13) + \varepsilon(t), \end{aligned}$$

where $\{\varepsilon(t)\}$ is a purely random process; that is, the $\varepsilon(t)$ are independent zero mean variates. Then it is easy to check that the covariance $\mathrm{E}[x(t)x(t+m)] = 0$ for all $m \neq 0$. Thus $\{X(t)\}$ is a stationary white-noise process with a nonlinear structure that yields dependences over many time points. A plot of the aforementioned time series, with the $\varepsilon(t)$ series having iid mean 0 normal distributions with variance $\sigma_\varepsilon^2 = .0003$, is presented in Figure 1. It was designed to approximate a certain stock-price time series with $\sigma_x^2 = .00058$. Even though the series is stationary, the time-path behavior when modeled in a linear time series fashion might lead an analyst to conclude erroneously that the series was nonstationary, with time-varying variance. In fact, if one divided the 2,000-point record from Figure 1 into 20 subintervals, calculated the variance for each subinterval, and applied standard statistical methods in a pro forma manner, then one would statistically reject the hypothesis of equal variances. The value of Bartlett's test for equal variances (df = 19) is 310.2, indicating

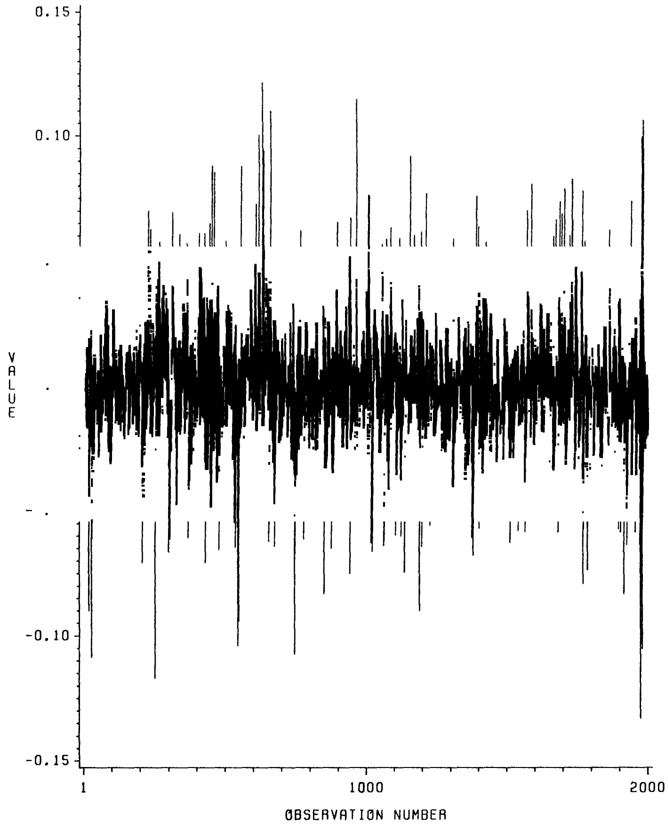

Figure 1. Nonlinear Stationary Time Series That Could Be Mistaken for a Nonstationary Series With a Time-Varying Variance if Modeled Linearly.

great departure from the equal-variance hypothesis (a 1% level of significance value for this test statistic is 36.2). In addition, if one examined the individual variances pairwise by using an F ratio, one would find that 101 of the 190 possible pairs show significance at the 1% level, and 121 of the 190 at the 5% level, again seemingly indicating a significant time-varying variance. Thus nonlinear stationary structure can be mistaken for nonstationarity when forced into a linear format.

Nonlinear structure has been detected in a variety of scientific time series data records using bispectral analysis (a term coined by Tukey 1959). Hasselman, Munk, and MacDonald (1963) analyzed nonlinear interaction of ocean waves in shallow water. MacDonald (1963) presented results on nonlinear interactions in atmospheric pressure data. Sato, Sasaki, and Nakamura (1977) used the bispectrum to analyze acoustic gear noise. The first application of the bispectrum to economic time series is credited to Godfrey (1965). Huber, Kleiner, and Gasser (1971) analyzed brainwave data for nonlinear interactions. The most thorough and statistically sophisticated application of the bispectrum to the analysis of physical data was presented in three papers by Lii, Rosenblatt, and Van Atta 1976, Van Atta 1979, and Helland, Lii, and Rosenblatt 1979. These papers use estimated bispectra to study nonlinear spectral transfer of energy in turbulence. Nonlinear energy transfers in plasmas were investigated through bispectral techniques by Kim and Powers (1978).

The bispectrum is the third-order cumulant spectrum.

Hinich and Clay (1968) provided some intuitive understanding of the bispectrum in terms of cross-frequency phase coherence. Rosenblatt (1983) gave a review of cumulant spectra and the asymptotic properties of their estimators. Subba Rao (1983) gave some practical considerations for bispectral estimation and discussed the relationship between the bispectrum and a certain class of nonlinear models called "bilinear" models.

Although the relationship between the structure of a nonlinear filter and the bispectrum of the filter output has been understood for at least 25 years, statistical and computational problems have severely limited progress. Computer software for bispectral analysis are not part of standard statistical software packages. The published papers on bispectral analysis do not give bispectral estimation routines. Subba Rao (1983) does not go into sufficient detail in his review of bispectral estimation to enable most readers even to organize a computational flowchart. A FORTRAN program is available from Patterson (1983), however, and we use it in this article. The monograph by Subba Rao and Gabr (1984) also contains a listing in FORTRAN for doing bispectral analysis (although they used mathematical subroutines from England).

Most applications do not involve a theoretically based model that indicates how well a linear time series model will approximate the data, so this must be determined from the data. Moreover, an easily applied computer-based method for detecting nonlinearity is important for the development of nonlinear analysis of time series data. When physical models indicate that nonlinear models should be used to analyze data, as is the case for turbulence in ocean waves or plasmas, then bispectral analysis is needed.

Another problem that has inhibited the application of bispectral analysis is the lack of statistical tests for significance for bispectral estimates. Even more important has been the absence of test statistics for detecting nonlinearity in time series data. Recently several tests for nonlinearity have been proposed, however, both in the frequency domain and the time domain. In the frequency domain, Subba Rao and Gabr (1980) were the first to implement Brillinger's (1965) method for measuring the departure of a process from linearity (and Gaussianity) by using an estimate of the bispectrum of the observed time series. Their tests do not use the asymptotic sampling properties of the bispectrum that were developed by Rosenblatt and Van Ness (1965), Shaman (1965), and Brillinger and Rosenblatt (1967 a,b), nor did they give test statistics for the significance of individual bispectral estimates. Hinich (1982) presented a streamlined and practical bispectral procedure for testing whether time series data are consistent with a linearity (and also a Gaussianity) hypothesis. The Hinich method is incorporated in the Patterson computer program, along with chi-squared statistics for testing the significance of the individual bispectral estimates. Ashley, Patterson, and Hinich (1986) showed that the Hinich tests have considerable power for detecting quadratic moving average and bilinear models for sample sizes as "small" as 256. In the time domain, several tests were proposed, including those of Keenan (1985), Petruccelli and Davies (1986), Granger and Newbold (1976), Maravall (1983), McLeod and Li (1983), Chan and Tong (1986), and Robinson (1983). These tests were compared and reviewed by Petruccelli and Davies (1986), Davies and Petruccelli (1985), and Chan and Tong (1986).

Unfortunately, linear models are often applied in a pro forma manner. Of course, if there are not enough data to measure the magnitude of a nonlinear structure in the process, then a linear model is the only reasonable approach. If there are enough data to test linearity in the residuals of the linear fit, then the investigator can decide whether to incorporate nonlinearity, based on appropriate test statistics. A global test statistic for linearity, such as the Hinich or Subba Rao tests, can detect the presence of measurable nonlinearity, just as the Durbin–Watson statistics help detect measurable serial correlation in a linear time series model.

Section 2 presents a sketch of the theory and methods for testing for linearity using a sample polyspectrum. Section 3 indicates how the sample bispectrum or the sample third-order cumulant has been used in various time series of real data to determine linearity and Gaussianity of the series.

2. STATISTICAL TESTS FOR LINEARITY AND GAUSSIANITY OF TIME SERIES

Subba Rao and Gabr (1980) and Hinich (1982) gave statistical tests for globally determining whether an observed stationary time series $\{X(n)\}$ is linear. It is possible that $\{X(n)\}$ is linear without being Gaussian, but all of the stationary Gaussian time series are linear. Both articles also included time series tests for joint Gaussianity, based on the sample bispectrum of the time series. The Hinich test is nonparametric and robust. Accordingly, our tests use the Hinich test.

Let $\{X(n)\}$ be a stationary time series and assume, without loss of generality, that $\mathrm{E}[X(n)] = 0$. The spectrum of $\{X(n)\}$ is the Fourier transform of the autocovariance function $C_X(n) = \mathrm{E}[X(t + n)X(t)]$,

$$S(f) = \sum_{n=0}^{\infty} C_X(n)\exp\{-2\pi i f n\}.$$

Many papers use the spectrum $S(f)$ as a way to examine the correlation structure of $X(n)$. [See Granger and Morgenstern (1963) for many applications of spectral analysis techniques to finance.] In particular, $X(n)$ is serially uncorrelated (white noise) if $S(f)$ is constant.

The *bispectrum* of $\{X(n)\}$ is defined as the (two-dimensional) Fourier transform of the third-moment function

$$B(f_1, f_2) = \sum_m \sum_n C_{XX}(n, m)\exp\{-2\pi i f_1 n - 2\pi i f_2 m\},$$

where $C_{XX}(n, m) = \mathrm{E}[X(t + n)X(t + m)X(t)]$. A rigorous introduction to the bispectrum and its symmetries and properties was given by Brillinger and Rosenblatt (1967 a,b). For our purposes, however, the bispectrum is important because it allows a statistical test for linearity (and also Gaussianity) of a time series, and for the significance of the individual bispectral estimates.

Suppose that $X(n)$ is a linear time series; that is, it has the form of Equation (1.1). Then it can be shown that the spectrum of $\{X(n)\}$ is of the form

$$S(f) = \sigma_\varepsilon^2 |A(f)|^2, \tag{2.1}$$

and the bispectrum of $\{X(n)\}$ is of the form

$$B(f_1, f_2) = A(f_1)A(f_2)A^*(f_1 + f_2)\mu_3, \tag{2.2}$$

where $\mu_3 = E[\varepsilon^3(t)]$, and $A(f)$ is the transform of the coefficient series

$$A(f) = \sum_{n=0}^{\infty} a(n)\exp(-2\pi i f n),$$

where A^* denotes the complex conjugate of A.

From Equations (2.1) and (2.2), it follows that

$$\frac{|B(f_1, f_2)|^2}{S(f_1)S(f_2)S(f_1 + f_2)} = \frac{\mu_3^2}{\sigma_\varepsilon^6} \tag{2.3}$$

is constant over all frequency pairs (f_1, f_2) if $\{X(n)\}$ is linear. If we define the class $\mathcal{T}$ to be the collection of all linear processes with innovation variables $\varepsilon(t)$ possessing $\mu_3 = 0$, then the ratio (2.3) is 0 for any time series $X(t) \in \mathcal{T}$. The class $\mathcal{T}$ contains the class $\mathcal{P}$ of all linear processes $X(t)$ with innovation variables $\varepsilon(t)$ symmetrically distributed, which in turn contains the class $\mathcal{N}$ of all Gaussian time series.

The relationship in Equation (2.3) is the basis of the Hinich tests. Since the bispectrum is a spatially periodic function whose values in the plane are completely determined via symmetry relations by the principal domain $\{(f_1, f_2) : 0 < f_1 < \frac{1}{2}, f_2 < f_1, 2f_1 + f_2 < 1\}$, Hinich constructs an estimate $\hat{B}$ of the bispectrum $B(f_1, f_2)$ and $\hat{S}$ of the spectrum $S(f)$. He then estimates the ratio in Equation (2.3) at different frequency pairs (f_1, f_2) in the principal domain by $|\hat{B}(f_1, f_2)|^2/\hat{S}(f_1)\hat{S}(f_2)\hat{S}(f_1 + f_2)$. If these ratios differ too greatly over different frequency pairs, he rejects the constancy of the ratio, and hence the linearity of the time series $\{X(n)\}$. If the estimates differ too greatly from 0, he rejects the hypothesis that the time series model belongs to class $\mathcal{P}$. In particular, Gaussianity is rejected if the ratio in (2.3) differs too greatly from 0. The constant $\mu_3^2/\sigma_\varepsilon^6$ is the square of Fisher's skewness measure for the ε series.

The test statistic Hinich derives for testing linearity is based on the inner quartile range of the estimated ratio over the set of pertinent frequency pairs. If the ratio in (2.3) is constant, then the inner quartile range is small; if it is not constant, then the inner quartile range is larger. The test statistic for linearity is asymptotically normal, so significance is readily determined from standard normal tables. See Hinich (1982) for the precise formulas and proofs concerning this test for linearity.

The test for the time series belonging to class $\mathcal{P}$ (and in particular a test for Gaussianity) involves testing whether the estimated ratio in (2.3) deviates only randomly from 0. Hinich (1982) derived an asymptotically normal test statistic based on the estimated ratio in (2.3) in this situation as well.

It should be emphasized that the time series under study can be serially uncorrelated and still fail to be either linear or Gaussian. For economic data generated by a form of idealized market structure known as an "efficient market," such a result might be expected, since such series appear to be white-noise series.

In Section 3 we show the results of applying the Hinich tests to several different data sets.

3. EXAMINATION OF LINEARITY AND GAUSSIANITY FOR REAL TIME SERIES

In this section we present a summary of the results of analyzing several different real data sets using the preceding statistical tests. The Hinich tests yield standard normal variates for the test statistic if the hypothesized time series model is indeed true. A one-tailed 1% level of significance for this statistic is 2.326.

To provide a benchmark for comparison, the first series examined was a sequence of values generated by a random-number generator that simulates Gaussian white noise. The series was divided into 100 records, each containing 1,024 samples. Each record was tested for linearity and Gaussianity. The averaged asymptotically standard normal test statistic values for the 100 records are $z = -3.3$ for the Gaussianity test and $z = -3.4$ for the linearity test. Since the test statistics are one-sided, only large positive values are significant; hence this series can easily be accepted as linear and Gaussian. The normalized bispectrum for each record was averaged over all records (see Fig. 2). The time unit is set equal to 1 second so that the bandwidth is 512 hertz. Except for small random perturbations, the graph appears flat (as it should).

3.1 Underwater Acoustical Sonar Time Series

Optimal detection of signals in stochastic noise has been developed in relatively few cases. Most investigators assume that the signal or the noise (or both) are Gaussian processes. Even when non-Gaussianity is allowed, the process is often assumed to be linear, or a simple martingale derived from a Gaussian process (e.g., via simple stochastic differential equations). Examples of this include the results presented by Brockett (1984a,b) and Baker and

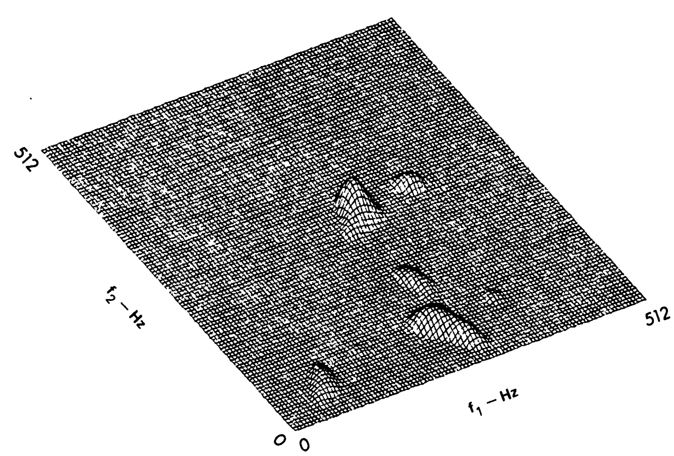

Figure 2. Normalized Bispectral Plot of Gaussian White Noise.

Gaultierotti (1984). In many situations, such as sonar, radar, or satellite transmission, the Gaussianity assumption may not hold (e.g., see Girodan and Haber 1972; Kennedy 1969; Middleton 1967a,b, 1972a,b; Trunk and George 1970; VanTrees 1971). It is thus of some interest to determine if signals and/or noise are Gaussian processes in the ocean environment. If they are non-Gaussian, we wish to determine if they can be modeled as non-Gaussian linear processes (e.g., autoregressive, moving average, or simple stochastic integrals). This is useful for determining which type of signal detector to implement for best performance. In some environments, man-made noise (e.g., merchant-shipping noises) or environmental noises (e.g., biological noises or ice-cracking noises) raise questions about the linearity or Gaussianity of the ambient noise field. The issue of linearity and Gaussianity of underwater acoustical ambient noise series was examined by Brockett, Hinich, and Wilson (1987). Their results shed some new and interesting light on the traditional assumption of Gaussianity of uncontaminated ambient noise fields. If the series are both nonlinear and non-Gaussian, then new nonlinear models must be developed.

The acoustical series examined in this article involves ambient noise contaminated with a biologically generated noise (snapping shrimp). Previous examination of this series determined that the marginal distributions were non-Gaussian (Wilson and Powell 1984); this fact is corroborated by the Hinich test statistic values calculated in Table 1. Nevertheless, more information is available using our bispectral tests. Not only is the observed time series shown to be non-Gaussian, it is nonlinear as well.

3.2 Common-Stock Price Series

Academic interest in securities markets tends to focus on the relationship of these markets to the ideal markets of economic theory, which asserts that competition in speculative markets reduces the expected economic gain to 0. This theory has led to mathematical models of security-price behavior that are often based on the assumption that the stochastic process of prices should have independent increments. As a consequence, empirical research in this field has concentrated on identifying the statistical process that generates security prices and ascertaining whether this process can be used to forecast prices. The consensus of published empirical research is that a geometric random walk describes prices fairly well, although some anomalies have been reported. We now apply the previously described time series tests to realizations of daily log-price relatives [i.e., $\ln(p_t/p_{t-1})$] for certain common stocks. These results challenge the belief that daily rates of return can be viewed as independent random variables or even as linearly smoothed independent random variables. [The rate of return, $r_t = (p_t - p_{t-1})/p_{t-1}$, is approximately equal to the log of the stock-price relative $\ln(p_t/p_{t-1})$ for small r_t.]

There is still considerable controversy about the probability model responsible for the generation of returns. At the risk of oversimplification, the explanations can be divided into two groups. In the first group, returns are generated by a stationary, non-Gaussian distribution, such as a member of the stable Paretian family. [Mandelbrot (1963, 1967) and Fama (1965) offered evidence supporting the stable Paretian model; Press (1967) argued for a homogeneous Poisson jump process superimposed on a Gaussian process; and Blattberg and Gonedes (1974) suggested Student's t distribution.] In the second group, returns are generated by a Gaussian model with nonstationary parameters. [Clark (1973) considered a normal distribution with a lognormal distribution for the variance parameter, whereas Hsu (1972) and Hsu, Miller, and Wichern (1974) presented evidence inconsistent with a stable Paretian model, and suggested a normal process with random jumps in the variance occurring at discrete points in time.] Table 2 summarizes the results of applying the bispectral tests to 10 different common-stock series. These results reinforce the results of Hinich and Patterson (1985) and support a third alternative: Daily returns are realizations of a nonlinear random process. Although stock-return time series closely resemble white noise, this evidence suggests a much higher degree of dependence in daily stock returns. In fact, the time-path behavior of nonlinear processes is such that the Hsu (1972) and Hsu, Miller, and Wichern (1974) results can be consistent with a variety of nonlinear processes (see Fig. 1). As can be seen from Table 2, the series of stock-price relatives is decidedly non-Gaussian and nonlinear.

A more detailed examination of the common-stock nonlinearities involves determining the extent to which the different stocks exhibit similar structural nonlinearities. Thus for each stock series we examined the individual frequency pairs that deviate significantly from the linearity hypothesis. The 1% level of significance for the χ^2 distribution with 2 df is approximately 10, and hence we assess the similarity of nonlinear structure by extracting for each stock those frequency pairs (f_1, f_2) whose normalized bi-

Table 1. Underwater Acoustical Sonar Series

Series	Gaussianity test statistic (z)	Linearity test statistic (z)
Biological noise (snapping shrimp)	89.7	87.6

Table 2. Summary of Gaussianity and Linearity Tests For Stocks

Firm	Gaussianity test statistic (z)	Linearity test statistic (z)
American Airlines	4.21	4.35
Alberto Culver Company	12.08	6.51
Columbia Broadcasting	4.71	4.09
Campbell Soup	13.39	5.48
El Paso Natural Gas	30.19	5.53
Swift & Company	22.81	7.83
Federated Department Stores	4.81	4.26
Northern Natural Gas	8.62	6.97
Indianapolis Power and Light	8.09	7.02
Merrill Lynch Pierce Fenner	6.54	1.85

NOTE: For all of the runs, in Hinich's (1982) notation the bispectral smoothing width was chosen as $M = 20$, and the width of the smoothing triangle for spectral estimation was chosen as $\Delta = 251$.

spectral value exceeds 10. The concordance across stocks is exhibited by dividing the frequency axes into subintervals, and for each subinterval counting the number of stocks with a χ^2 value exceeding 10. Figure 3 displays a bifrequency plot of the significant pairs for the 10 stocks. As can be seen, the stocks exhibit similarity in the location of their significant nonlinear frequency interactions, and hence the nonlinear time series structures found in Table 2 have some commonality across stocks. This evidence gives credence to the hypothesis that similar market-structure mechanisms (e.g., programmed institutional buying and selling) cause the discovered nonlinear time series behavior of common-stock returns.

3.3 The Spot and Forward Foreign Exchange-Rate Series

There have been many articles examining the spot-price series, the forward-price series, and the relationship between the spot- and forward-price series for foreign exchange rates. Some authors attempt to specify formally the form of a market equilibrium by postulating a particular linear (often regression) relationship between the spot price and the forward price. The residual error terms, or sometimes the forecast errors, are assumed to be iid normal variates (white Gaussian noise), or perhaps serially correlated linear processes; hypothesis tests about the values of certain parameters in the model are used to infer market efficiency, exchange-rate bias, risk premiums for trading, or other characteristics of interest. Linear models

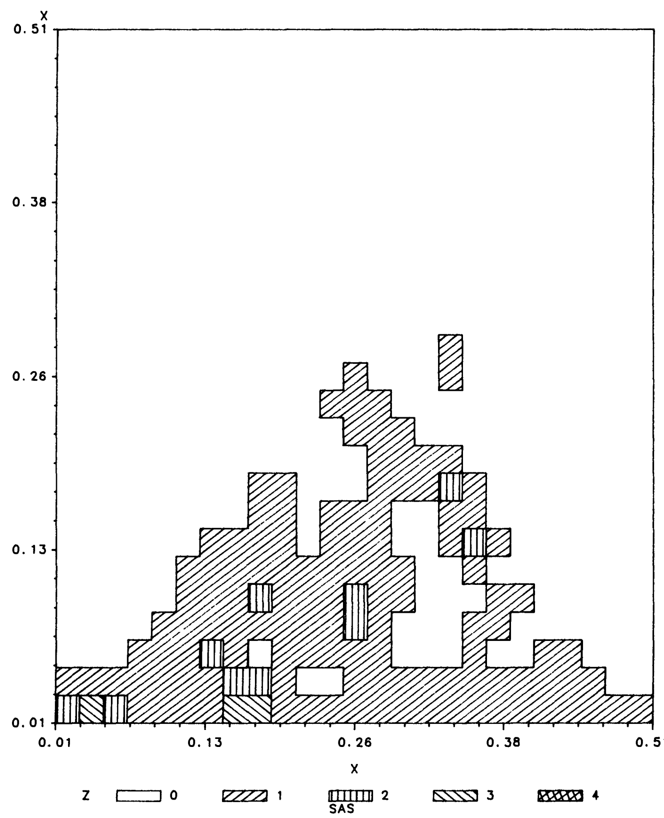

Figure 3. Bispectrum Results for 10 Stocks. The plot shows the number of stocks with a significant bispectrum for each frequency pair. For each data set, M = 20, N = 249, and there are 1,000 observations.

with normally distributed errors are used for statistical convenience. It does not follow from economic principles that appropriate models for exchange rates must be linear processes with Gaussian white-noise residual errors.

From the perspective of the exchange-rate series themselves, many articles support the random-walk hypothesis for foreign exchange rates. Among these papers are those by Giddy and Dufey (1975), Callen, Kwan, and Yip (1985), Mussa (1979), Fama (1983), Hodrick and Srivastava (1983), and Korajcyk (1983). These researchers did not have access to newly developed statistical tests for linearity of time series, so in using the well-developed theory based on linear Gaussian processes they lacked the ability to check their data for compatibility with this assumption. In this section we analyze foreign exchange rates and find that the linear models previously postulated (and in particular the random-walk hypothesis) do not fit the data. Let $S(t)$ denote the spot exchange price at time t; $F(s, t - s)$ is the corresponding forward price at time $t - s$ looking forward to predict the spot price s periods later. We use 30-day forward prices; let a period be 30 days. In addition to the examination of the original-price time series, there have been other contending models involving rates (log price) rather than the prices themselves. Thus we are led by the literature on foreign exchange rates to examine additionally the time series $\ln S(t)$ and $\ln F(s, t - s)$. Of course, modeling a linear (random walk) relationship for rates is equivalent to a log-linear (geometric random walk) relationship for prices, and the geometric random walk is a model often used in finance research.

In applying the preceding statistical tests to the analysis of the forward and spot time series for foreign exchange rates, we examined both the original price quotes and the log of the price (the rate). For analysis we chose the U.S. dollar to Japanese yen exchange rate, since this is ostensibly one of the most closely watched and tightly arbitraged currency exchanges. (If linearity and/or Gaussianity is to be found in foreign exchange rates, this is a likely place to find it.) We examined two time periods, from January 1, 1981, to mid-1982 and from December 12, 1981, to mid-1983. We took daily quotes for rates from *The Wall Street Journal*, using the 30-day forward rate $F(30, t)$ and the corresponding spot rate $S(30 + t)$. When data points are missing (e.g., trading holidays and weekends) the daily values are imputed with linear interpolation on the existing data.

Table 3 shows the results of the analysis applied to the spot, log-spot, forward, and log-forward time series. The Hinich tests yield standard normal variates for the test statistic if the hypothesized time series model is indeed a linear and/or Gaussian series. As can be seen immediately, the spot, forward, log-spot, and log-forward time series are nonlinear and do not belong to class $\mathcal{P}$ (and hence are also non-Gaussian). Since most of the current models used in foreign exchange-rate studies are all incompatible with nonlinear time series, the conclusions previously drawn by these authors must be reexamined in light of this new statistical evidence.

Table 3. Results of Test for Foreign Exchange-Rate Time Series Linearity and Gaussianity

Original series	Test period	Gaussianity test statistic (z)	Linearity test statistic (z)
$S(t)$	1/2/81–mid-'82	104.39	232.75
	12/31/81–mid-'83	20.73	17.67
$F(30, t)$	1/2/81–mid-'82	119.56	350.21
	12/31/81–mid-'83	28.07	25.42
$\ln S(t)$	1/2/81–mid-'82	25.78	9.98
	12/31/81–mid-'83	6.69	2.70
$\ln F(30, t)$	1/2/81–mid-'82	164.67	948.82
	12/31/81–mid-'83	3.91	.80

4. SUMMARY AND CONCLUSIONS

Linear models are usually applied in a pro forma manner. If there are not enough data to measure the magnitude of a nonlinear structure, then application of a linear model is the only reasonable approach. If there are enough data to test linearity in the residuals of the linear fit, then the investigator can decide whether to try to incorporate nonlinearity based on appropriate test statistics. A global test for linearity such as the Hinich or Subba Rao tests can detect the presence of measurable nonlinearity, just as the Durbin–Watson statistics help detect measurable serial correlation in a linear time series model, and residual plots help detect non-Gaussianity in iid stochastic models. In this article we have presented the Hinich tests for linearity and nonskewness (in particular Gaussianity) of time series and applied these results to obtain new information about both real time series that had previously been modeled linearly and also issues currently important in other disciplines. In some series the linearity model was found to be an acceptable time series model, and in other important series the tests found significant nonlinearities.

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