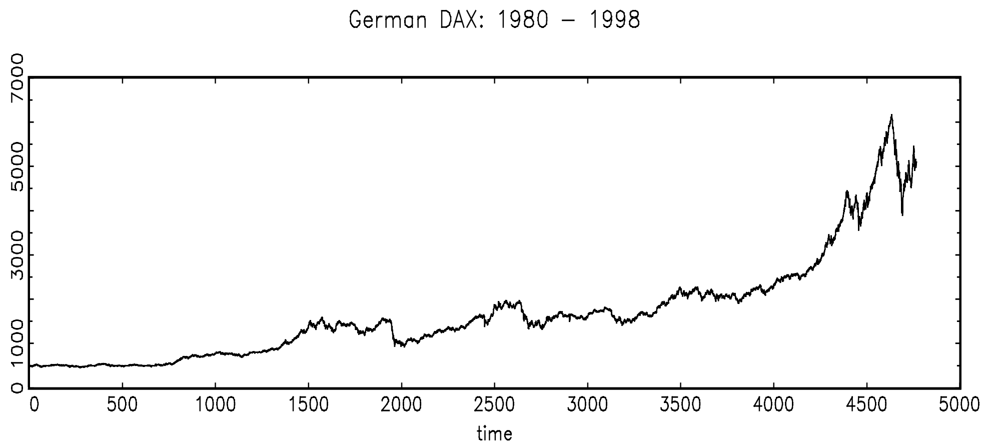
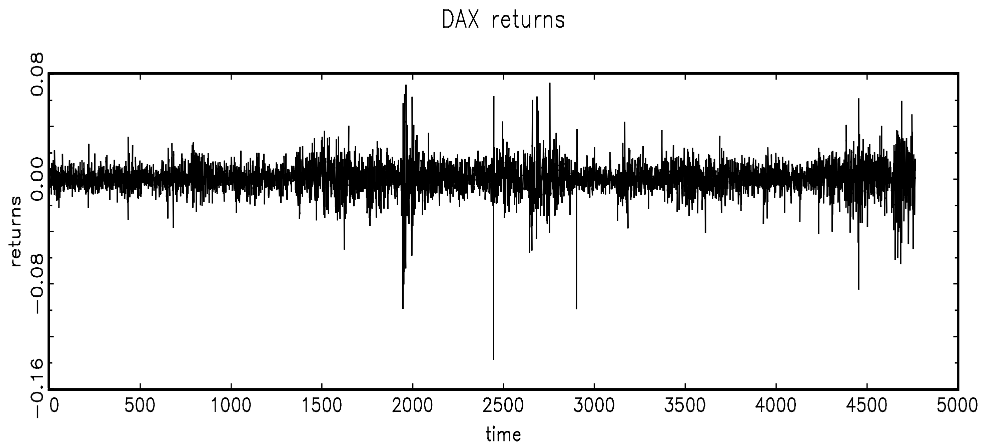
The stylized facts: some typical time series

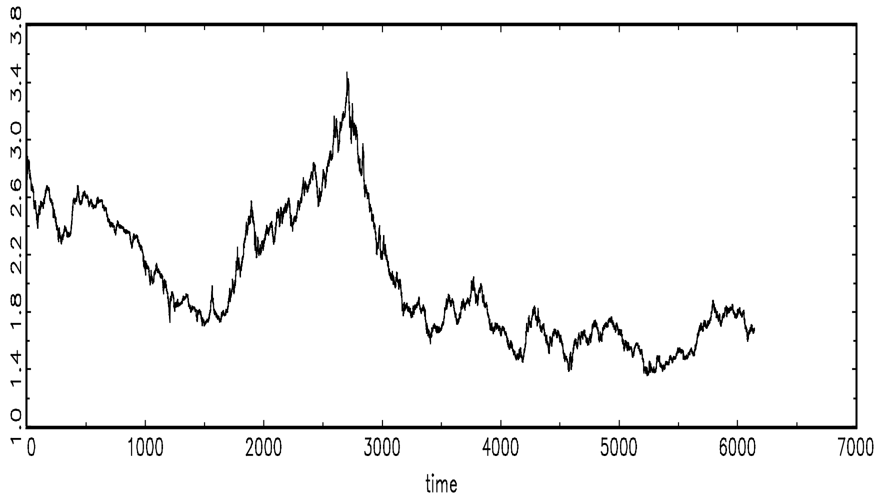


returns =

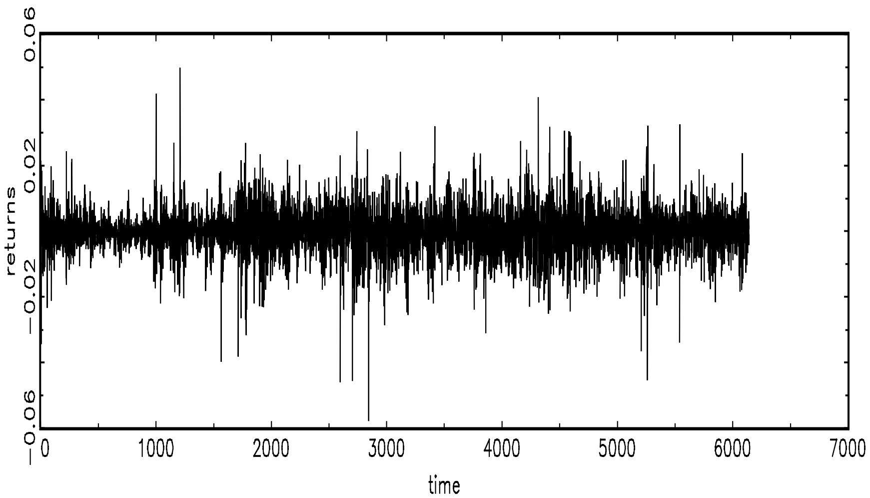
$\ln(p_t) - \ln(p_{t-1})$

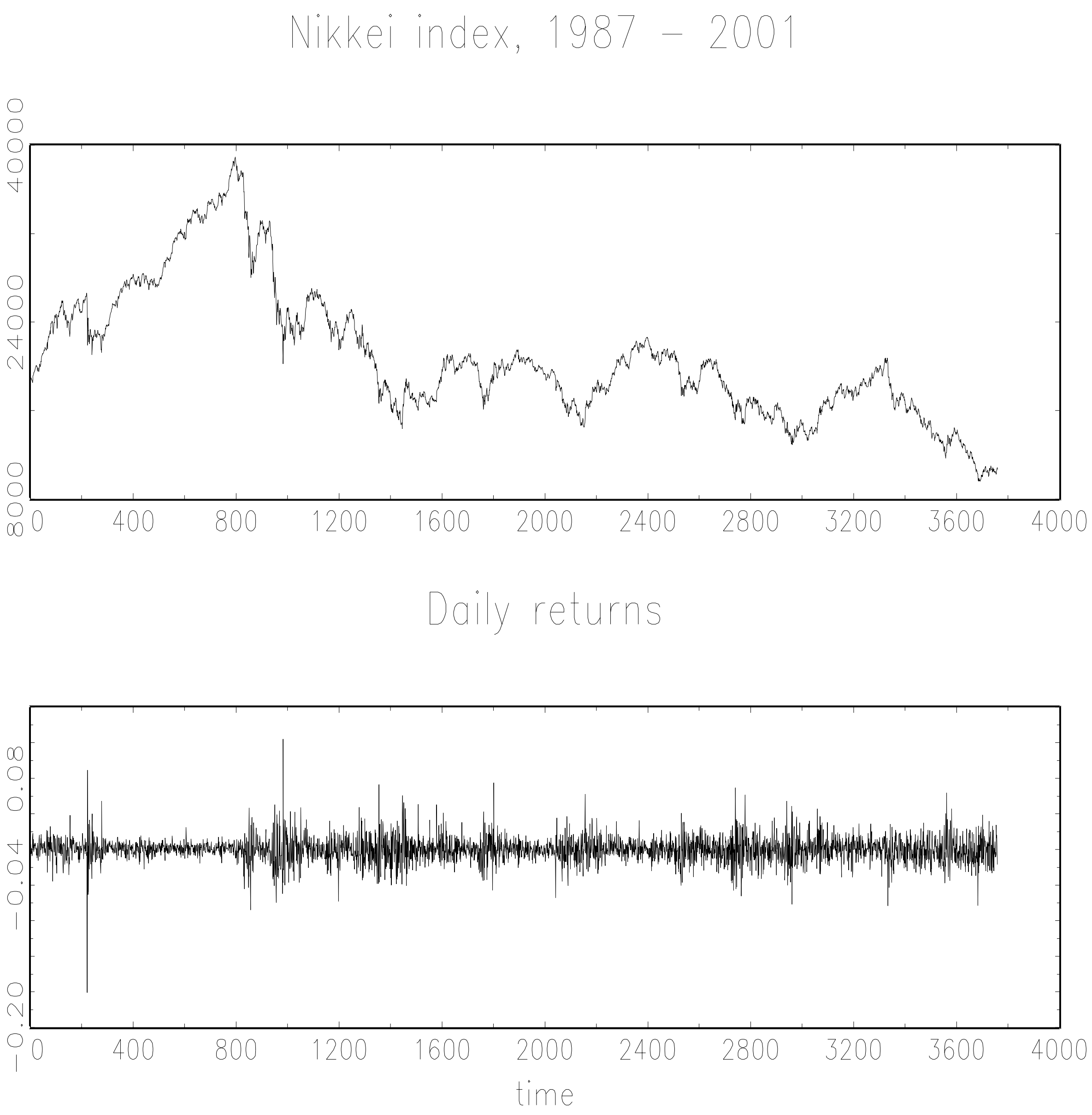


DM/U.S.$ exchange rate: 1974 – 1998



Relative daily changes of DM/U.S.$ exchange rate




Nikkei index, 1987 – 2001
8000
24000
40000
0
400
800
1200
1600
2000
2400
2800
3200
3600
4000
Daily returns
−0.20
−0.04
0.08
0
400
800
1200
1600
2000
2400
2800
3200
3600
4000
time

‘Stylized Facts’: Most Elementary and Universal Time Series Characteristics

- Random walk (martingal) property of prices
- Fat tails of returns
- Clusters of volatility
- Volume-volatility correlation

Fact 1: Random Walk (Martingal) Property of Prices

Prices (or their logs) look like realizations of a simple process:

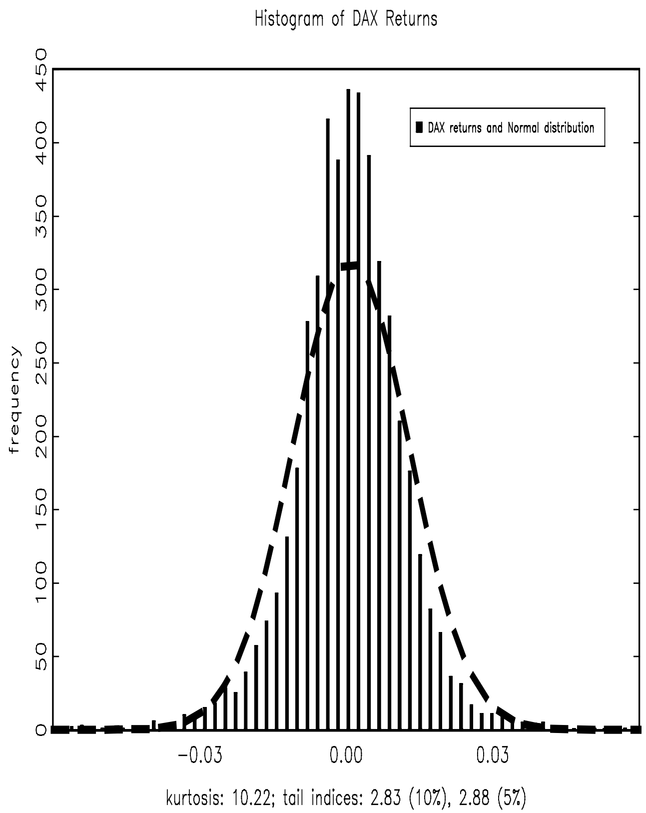
$$p_t = p_{t-1} + \varepsilon_t \qquad \text{with } E[\varepsilon_t] = 0$$

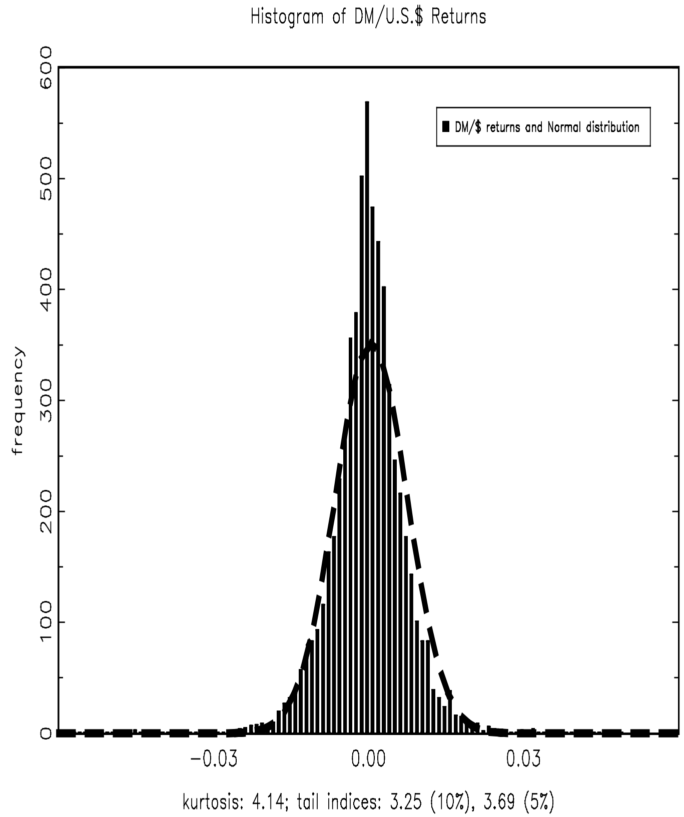
if ε_t is drawn from a fixed distribution, one speaks of a *random walk*, if no further assumptions are made other than $E[\varepsilon_t] = 0$, one speaks of a *martingal* (note that this allows for dependence in the variance ⇔ volatility clustering)

random walk (martingal) implies:

- absence of autocorrelations in returns (relative price changes),
- no predictability of future price developments,
- ***efficient*** price formation

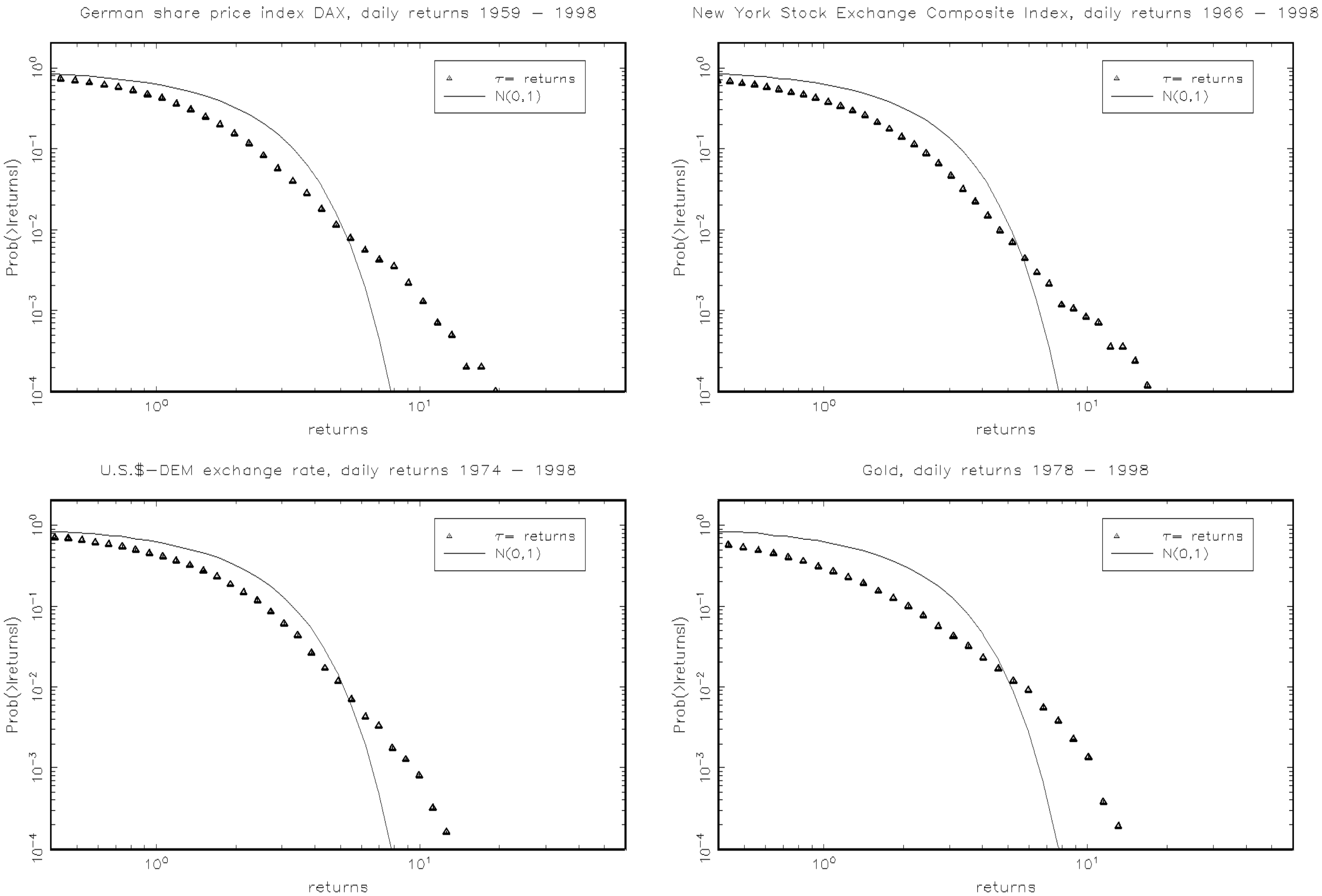
Fact 2: Fat Tails (Leptokurtosis) of the Distribution of Returns: Typical Histograms





Fat tails are more clearly visible in logarithmic representation of the histogram:

probability for large returns: Prob(ret. > x) ~ $x^{-\alpha}$ with $\alpha \in [2,5]$



Heavy Tails: History of Thoughts

- Daily returns exhibit ***leptokurtosis:*** probability of large realizations

  is ***always much*** higher than under a Normal distribution

- Puzzling, because returns are summable and should, therefore, follow the

  Central Limit Law -> convergence to Normality (Bachelier, 1900)

- Mandelbrot/Fama (1963): ***Generalized Central Limit Law***

  -> hypothesis: convergence to Levy Stable Distributions

- Recently emerging consensus: daily returns follow ***neither Normal nor Levy Stable*** distribution

- convergence to Normal only with low-frequency data (e.g., monthly returns)

Levy Stable Distributions

The so-called Levy Stable distributions are the *limiting distributions of sums of random variables.*

Drawback: no general analytical solution is known, Levy distributions can only be described by their characteristic function (only for illustration, you don‘t have to *learn* this formula):

$$\log E(e^{ixt}) = \begin{cases} i\delta t - |c\,t|^{\alpha_s}\,[1 - iß\,\mathrm{sgn}(t)\tan(\alpha_s/2)] & \text{if } \alpha_s \neq 1 \\ it - |c\,t|\,[1 + iß\,(2/\pi)\,\mathrm{sgn}(t)\log|c\,t|\,] & \text{if } \alpha_s = 1. \end{cases}$$

Parameters: ß, c, δ : parameters for skewness, scale and location,

α_s: characteristic exponent, $\alpha_s \in (0,2]$, parameter for *tail behavior*

$\alpha_s = 2$: Normal distribution, $\alpha_s < 2$: Levy distributions with *infinite variance*

Advantages: stability under aggregation, bellshape, leptokurtosis

Mandelbrot (1971): Most successful model in economics

Basics of Extreme Value Theory

The *tail behavior of most continuous distributions can be described by one of only three types of behavior:*

- *Finite endpoint (e.g., a uniform distribution over a finite interval)*
- *Exponential decline à la:* F(x) = 1- exp(-x) (e.g., the Normal distribution)
- *Power-law decline à la:* $F(x) = 1- x^{-\alpha}$ (e.g., the Student *t* distribution)

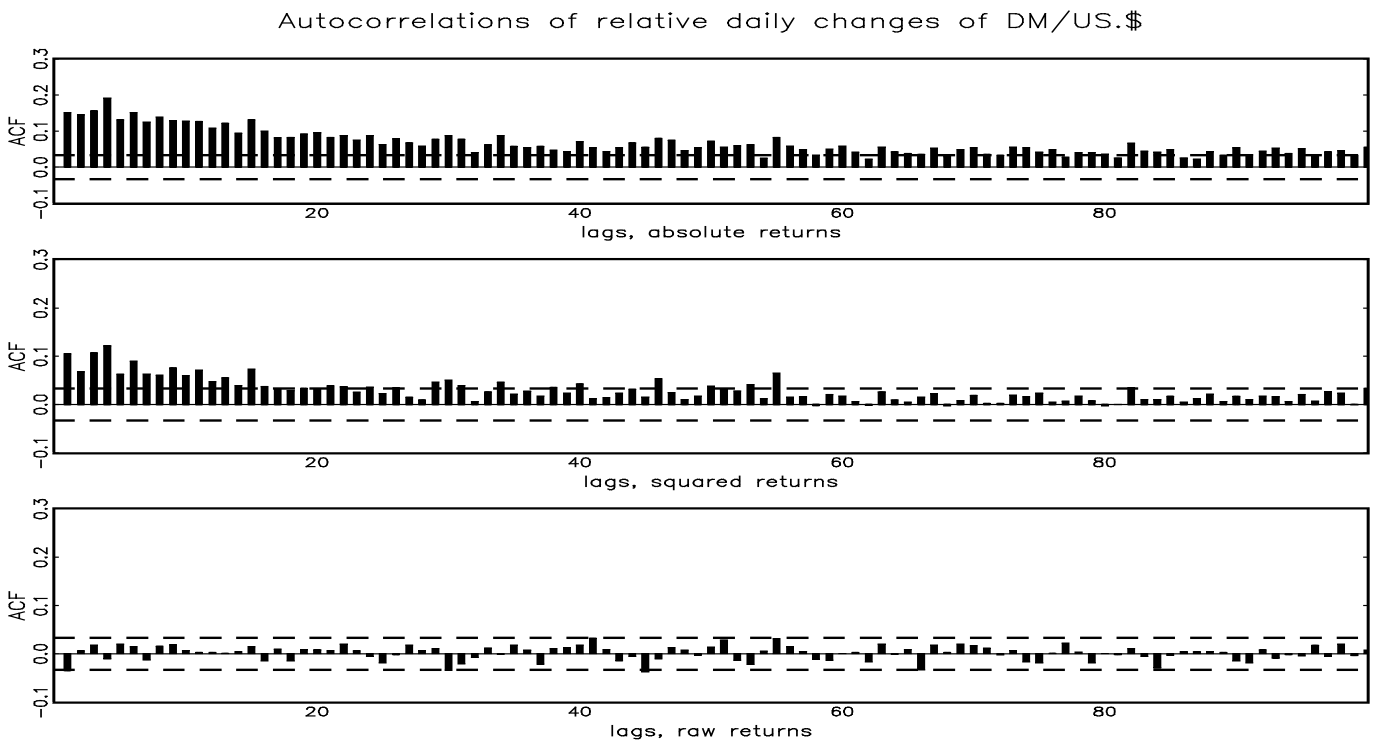
α: tail index (the lower, the slower the decay in the tail, i.e., the more large realizations occur)

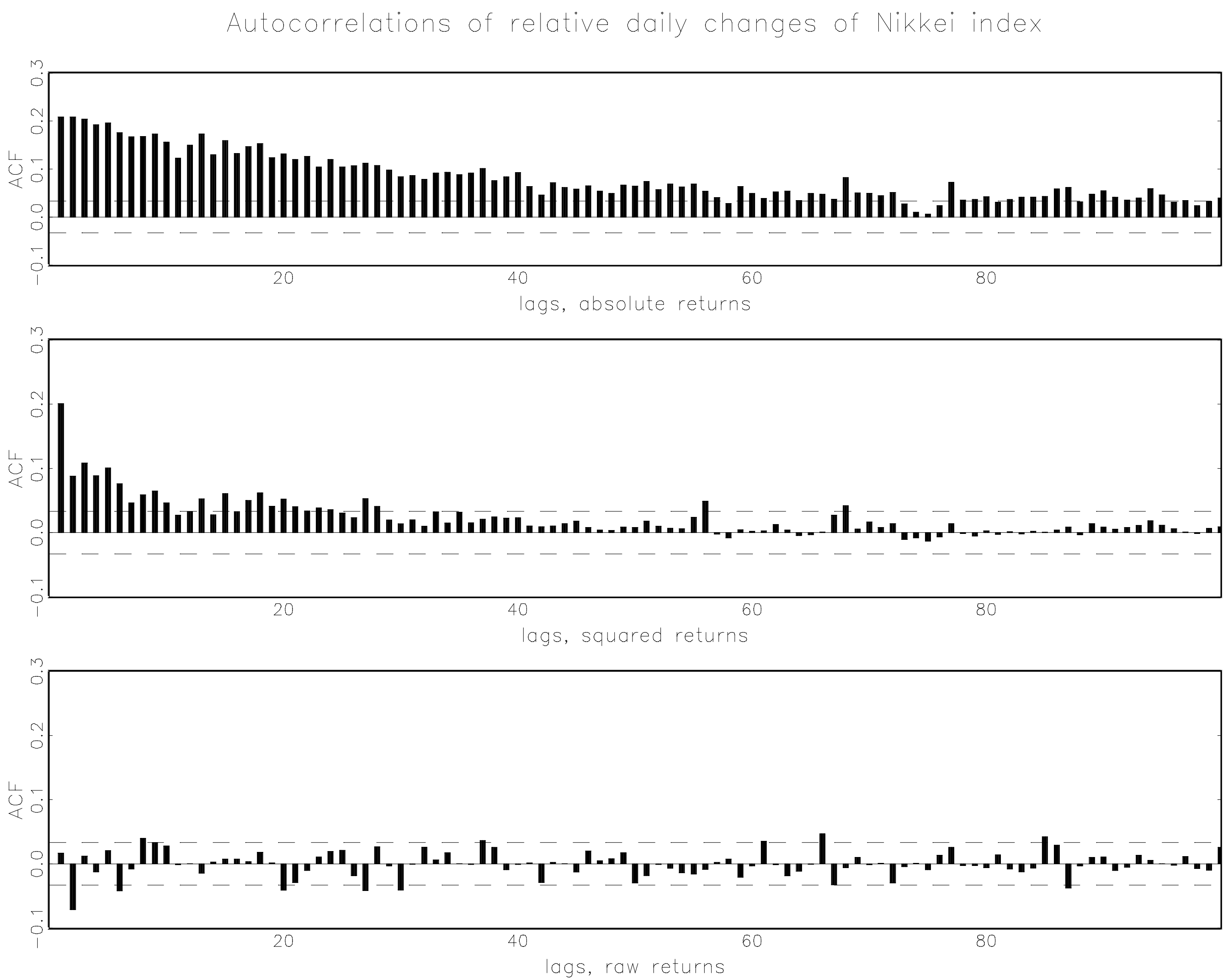
Empirical results for returns: power-law behavior with $\alpha \approx 3$ (cf. the values noted below the histograms)

Consequences: one rejects the Levy ($\alpha = \alpha_s \leq 2$) as well as the Normal distribution

Fact 3: Volatility clustering: ***autocorrelation in all measures of volatility***, e.g. ret^2, abs(ret) etc.

despite absence of autocorrelation in raw returns




Autocorrelations of relative daily changes of Nikkei index
ACF
lags, absolute returns
lags, squared returns
lags, raw returns
-0.1
0.0
0.1
0.2
0.3
20
40
60
80

The Baseline Stochastic Model for Volatility Clustering:

GARCH: Generalized Autoregressive Conditional Heteroscedasticity

returns follow: $r_t = \varepsilon_t$ with $\varepsilon_t \sim N(0, h_t)$

with $$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}$$ *(Engle, 1982; Bollerslev, 1986)*

Usual results:

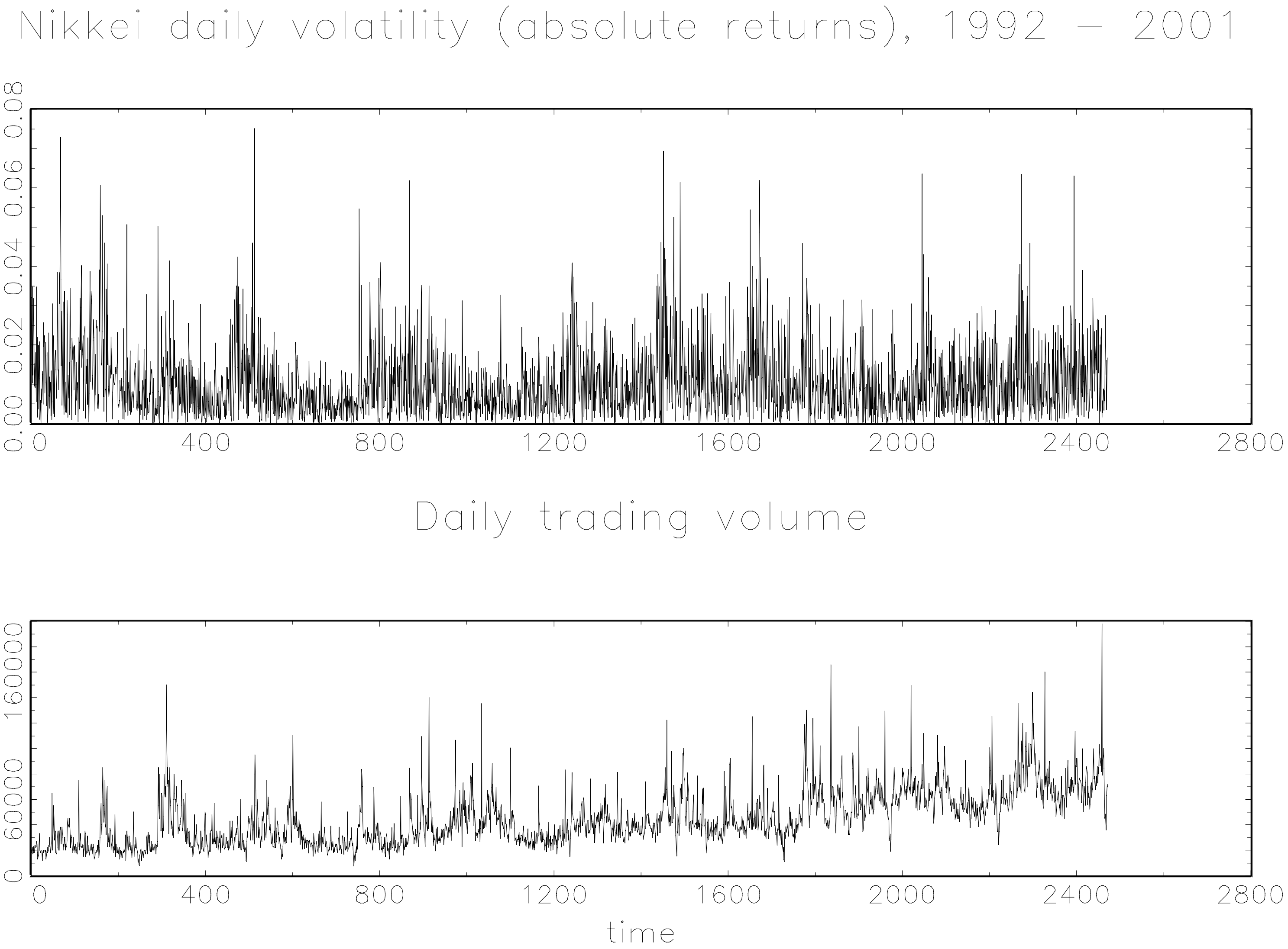
- GARCH(1,1) is sufficient to capture the volatility dynamics
- high persistence: $\alpha_1 \approx 0.1$, $\beta_1 \approx 0.9$
- process is close to the non-stationary case, i.e. $\alpha_1 + \beta_1 \approx 1$

Example: Typical GARCH Estimates for Daily Returns

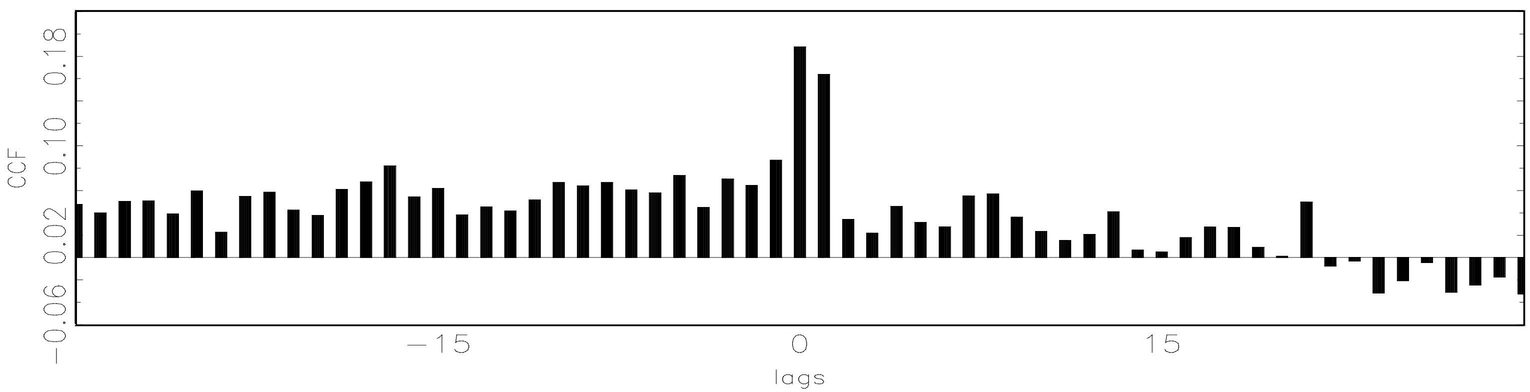
Data	α_0	α_1	β_1
DAX (1959 –1998)	$4.14*10^{-6}$ (5.07)	0.1520 (7.61)	0.8186 (49.86)
NYCI (1966 – 1998)	$1.04*10^{-6}$ (4.44)	0.0787 (3.81)	0.9093 (48.18)
US$-DM (1974 – 1998)	$8.50*10^{-7}$ (3.74)	0.1049 (8.26)	0.8836 (80.45)
Gold (1978 – 1998)	$1.14*10^{-6}$ (2.46)	0.1023 (6.78)	0.8853 (54.25)

t-values of the GARCH parameters are given in parenthesis.

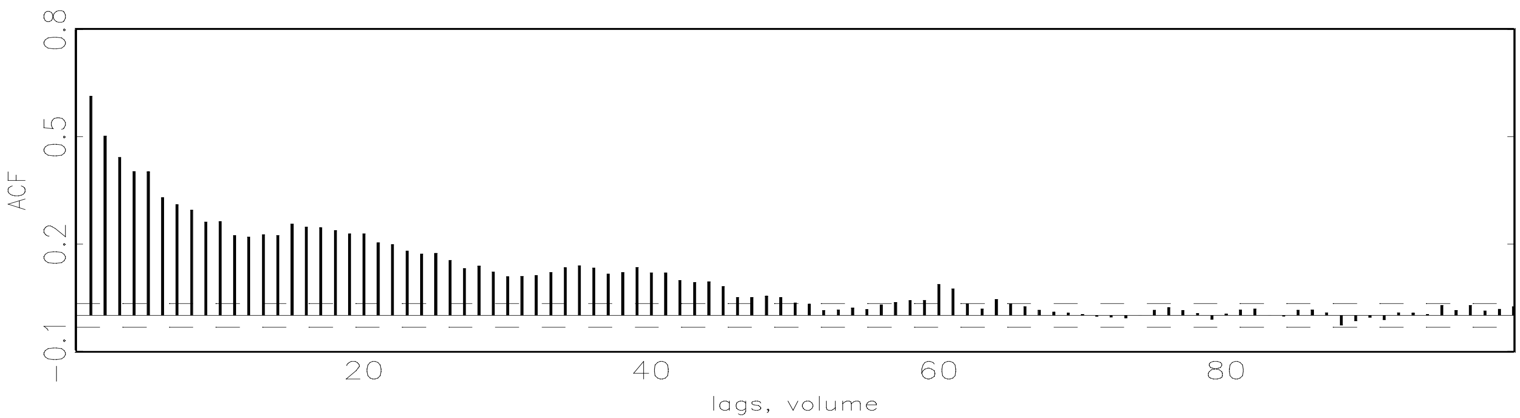
Fact 4: Correlation between Volatility and Volume



Crosscorrelation between Volume and Volatility: Nikkei, 1992 – 2001



Autocorrelation of Volume: Nikkei, 1992 – 2001 (detrended)



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