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INVESTMENT FOR THE LONG RUN: NEW EVIDENCE FOR AN OLD RULE

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I. BACKGROUND

"INVESTMENT FOR THE LONG RUN," as defined by Kelly [7], Latané [8] [9], Markowitz [10], and Breiman [1] [2], is concerned with a hypothetical investor who neither consumes nor deposits new cash into his portfolio, but reinvests his portfolio each period to achieve maximum growth of wealth over the indefinitely long run. (The hypothetical investor is assumed to be not subject to taxes, commissions, illiquidities and indivisibilities.) In the long run, thus defined, a penny invested at 6.01% is better—eventually becomes and stays greater—than a million dollars invested at 6%.

When returns are random, the consensus of the aforementioned authors is that the investor for the long run should invest each period so as to maximize the expected value of the logarithm of (1 + single period return). The early arguments for this "maximum-expected-log" (MEL) rule are most easily illustrated if we assume independent draws from the same probability distribution each period. Starting with a wealth of W_0, after T periods the player's wealth is

$$W_T = W_0 \cdot \prod_{t=1}^{T} (1 + r_t) \tag{1}$$

where r_t is the return on the portfolio in period t. Thus

$$\log(W_T / W_0) = \sum_{t=1}^{T} \log(1 + r_t) \tag{2}$$

If $\log(1 + r)$ has a finite mean and variance, the weak law of large numbers assues us that for any $\epsilon > 0$

$$\operatorname{Prob}\left(\left| \frac{1}{T} \cdot \log(W_T / W_0) - E\log(1 + r) \right| > \epsilon \right) \rightarrow 0 \tag{3}$$

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and the strong law[1] assures us that

$$\lim_{T\to\infty} \frac{1}{T}\cdot\log(W_T/W_0)=E\log(1+r) \tag{4}$$

with probability 1.0. Thus if $E\log(1+r)$ for portfolio A exceeds that for portfolio B, then the weak law assures us that, for sufficiently large T, portfolio A has a probability as close to unity as you please of doing better than B when time $=T$; and the strong law assures us that

$$W_T^A/W_T^B\to\infty \qquad \text{with probability one.} \tag{5}$$

Some authors have argued that the strategy which is optimal for the player for the long run is also a good rule for some or all real investors. My own interest in the subject stems from a different source. In tracing out the set of mean, variance (E, V) efficient portfolios one passes through a portfolio which gives approximately maximum $E\log(1+r)$.[2] I argued that this "Kelly-Latané" point should be considered the upper limit for conservative choice among E, V efficient portfolios, since portfolios with higher (arithmetic) mean give greater short-run variability with less return in the long run. A real investor might, however, perfer a smaller mean and variance, giving up return in the long run for stability in the short run.

Samuelson [14] and [15] objected to MEL as the solution to the problem posed in [1], [2], [7], [8], [9], [10]. Samuelson's objection may be illustrated as follows: suppose again that the same probability distributions of returns are available in each of T periods, $t=1,2,\ldots,T$. (Samuelson has also treated the case in which t is continuous; but his objections are asserted as well for the original discussion of discrete time. The latter, discrete time, analysis is the subject of the present paper.) Assume that the utility associated with a play of a game is

$$U=W_T^\alpha/\alpha \qquad \alpha\neq 0 \tag{6}$$

where W_T is final wealth. Samuelson shows that, in order to maximize expected utility for the game as a whole, the same portfolio should be chosen each period. This always chosen portfolio is the one which maximizes single period

$$EU=E(1+r)^\alpha/\alpha. \tag{7}$$

Furthermore, if EU_T^0 is the expected return provided by this strategy for a T

1. In most cases the early literature on investment for the long run used the weak law of large numbers. The results in Breiman [1], however, specialize to a strong law of large numbers in the particular case of unchanging probability distributions. See also the Doob [4] reference cited by Breiman.

2. Markowitz [10] Chapters 6 and 13 conjectures, and Young and Trent [16] confirm that

$$E\log(1+r)\approx\log(1+E)-\tfrac{1}{2}\cdot(V/(1+E)^2)$$

for a wide class of actual ex post distributions of annual portfolio returns.

period game, and EU_T^L is that provided by MEL, usually we will have

$$EU_T^0 / EU_T^L \to \infty \qquad \text{as } T \to \infty \tag{8}$$

Thus, despite (3), (4) and (5), MEL does not appear to be asymptotically optimal for this apparently reasonable class of games.

Von Newman and Morgenstern [17] have directly and indirectly persuaded many, including Samuelson and myself, that, subject to certain caveats, the expected utility maxim is the correct criterion for rational choice among risky alternatives. Thus if it were true that the laws of large numbers implied the general superiority of MEL, but utility analysis contradicted this conclusion, I am among those who would accept the conclusions of utility analysis as the final authority. But not every model involving "expected utility" is a valid formalization of the subject purported to be analyzed. In particular I will argue that, on closer examination, utility analysis supports rather than contradicts MEL as a quite general solution to the problem of investment for the long run.

II. The Sequence of Games

It is important to note that (8) is a result concerning a *sequence of games*. For fixed T, say $T = 100, EU = EW_{100}^{\alpha}/\alpha$ is the expected utility (associated with a particular strategy) of a game involving precisely 100 periods. For $T = 101, EW_{101}^{\alpha}/\alpha$ is the expected utility of a game lasting precisely 101 periods; and so on for $T = 102, 103, \ldots$.

That (8) is a statement about a sequence of games may be seen either from the statement of the problem or from the method of solution. In Samuelson's formulation W_T is *final* wealth—wealth at the *end* of the game. If we let T vary (as in "$T \to \infty$") we are talking about games of varying length.

Viewed differently, imagine computing the solution by dynamic programming starting from the last period and working in the direction of earlier periods. (Here we may ignore the fact that essentially the same solution reemerges in each step of the present dynamic program. Our problem here is not how to compute a solution economically, but what problem is being solved). If we allow our dynamic programming computer to run backwards in time for 100 periods, we arrive at the optimum first move, and the expected utility for the game as a whole given any initial W_0, for a game that is to last 100 moves. If we allow the computer to continue for an additional 100 periods we arrive at the optimum first move, and the expected utility for the game as a whole given any initial W_0, for a game that is to last for 200 moves; and so on for $T = 201, 202, \ldots$.

In particular, equation (8) is not a proposition about a single game that lasts forever. This particular point will be seen most clearly later in the paper when we formalize the utility analysis of unending games.

To explore the asymptotic optimality of MEL, we will need some notation concerning sequences of games in general. Let $T_1 < T_2 < T_3 \cdots$ be a sequence of strictly increasing positive integers. In this paper[3] we will denote by $G_1, G_2, G_3 \ldots$ a

3. A somewhat different, but equivalent, notation was used in [11].

sequence of games, where the ith game lasts T_i moves. (In case the reader feels uncomfortable with the notion of a sequence of games, as did at least one of our colleges who read [11], perhaps the following remarks may help. The notion of a sequence of games is similar to the notion of a sequence of numbers, or a sequence of functions, or a sequence of probability distributions. In each case there is a first object (i.e., a first number or function or distribution or game) which we may denote as G_1; a second object (number, function, distribution, game) which we may denote by G_2; etc.).

In general we will not necessarily assume that the same opportunities are available in each of the T_i periods of the game G_i. We will always assume that—as part of the rules that govern G_i—the game G_i is to last exactly T_i periods, and that the investor is to reinvest his entire wealth (without commissions, etc.) in each of the T_i periods. Beyond this, specific assumptions are made in specific analyses.

In addition to a sequence of games, we shall speak of a sequence of strategies $s_1, s_2, s_3, \ldots$ where s_i is a strategy (i.e., a complete rule of action) which is valid for (may be followed in) the game G_i. By convention, we treat the utility function as part of the specification of the rules of the game. The rules of G_i and the strategy s_i together imply an expected utility to playing that game in that manner.

III. Alternate Sequence-of-Games Formalizations

Let g equal the rate of return achieved during a play of the game G_i; i.e., writing T for T_i:

$$W_T = W_0 \cdot (1+g)^T \tag{9}$$

or

$$g = (W_T / W_0)^{1/T} - 1. \tag{10}$$

In the Samuelson sequence of games, here denoted by $G_1, G_2, G_3, \ldots$, the utility function of each game G_i was assumed to be

$$U = f(W_T) = W_T^{\alpha}/\alpha. \tag{11}$$

We can imagine another sequence of games—call them $H_1, H_2, H_3, \ldots$—which have the same number of moves and the same opportunities per move as $G_1, G_2, G_3, \ldots$, respectively, but have a different utility function. Specifically imagine that the utility associated with a play of each game H_i is

$$U = V(g). \tag{11a}$$

for some increasing function of g. For a fixed game of length $T = T_i$, we can always find a function $V(g)$ which gives the same rankings of strategies as does some specific $f(W_T)$. For example, for fixed T (11) associates the same U to each possible play as does

$$U = V(g) = W_0^{\alpha} \cdot (1+g)^{\alpha T}/\alpha. \tag{11b}$$

Thus for a given T it is of no consequence whether we assume that utility is a function of final wealth W_T or of rate of return g.

On the other hand, the assumption that some utility function $V(g)$ remains constant in a sequence of games, as in $H_1, H_2, H_3, \ldots$ has quite different consequences than the assumption that some utility function $f(W_T)$ remains constant as in $G_1, G_2, \ldots$. Markowitz [11] shows that if $V(g)$ is continuous then

$$EV_T^L / EV_T^0 \to 1 \qquad \text{as } T \to \infty \tag{12a}$$

where EV_T^L is the expected utility provided by the MEL strategy for the game H_i, and EV_T^0 is the expected utility provided by the optimum strategy (if such an optimum exists[4]); and if $V(g)$ is discontinuous then

$$EV_T^L / EV_T^0 \geqslant 1 - \epsilon - \delta \tag{12b}$$

where δ is the largest jump in V at a point of discontinuity, and $\epsilon \to 0$ as $T \to \infty$. (12a) and (12b) do not require the assumption that the same probability distributions are available each period. It is sufficient to assume that the return r is bounded by two extremes $\underline{r}$ and $\bar{r}$:

$$-1 < \underline{r} \leqslant r \leqslant \bar{r} < \infty \tag{13}$$

e.g., the investor is assumed to not lose more than, say, 99.99% nor make more than a million percent on any one move in any play of any game of the sequence. Note also that $V(g)$ is not required to be concave, nor strictly increasing nor differentiable; but of course it is allowed to be such.

Thus under quite general assumptions, if $V(g)$ is continuous MEL is asymptotically optimal in the sense of 12a. If $V(g)$ has small discontinuities, then MEL may possibly fail to be asymptotically optimal by small amounts as in 12b. These results are in contrast to (8), derived on the assumption of constant $U = f(W_T)$.

In [11] I argued that the assumption of constant $V(g)$ in a sequence of games is a more reasonable formalization of "investment for the long run" than is the assumption of constant $U(W_T)$. Given the basic assumptions of utility analysis, the choice between constant $V(g)$ and constant $U(W_T)$ is equivalent to deciding which of two types of questions would be more reasonable to ask (or determine from revealed preferences) of a rational player who invests for the long run in the sense under discussion.

Example of question of type I: what probability would make you indifferent between (a) a strategy which yields 6% with certainty in the long run: and (b) a strategy with a probability α of yielding 9% in the long run versus a probability of $1-\alpha$ of yielding 3% in the long run.

Example of question of type II: if your initial wealth is \$10,000.00, what

4. The assumptions of [11] do not necessarily imply that an optimum strategy exists. In any case (12a) and (12b) apply to any "other" strategy such that

$$EV_T^0 \geqslant EV_T^L \qquad \text{for all } T.$$

probability β would make you indifferent between (a) a strategy which yields \$20,000 with certainty in the long run, versus (b) a strategy which yields \$25,000 with probability β and \$15,000 with probability $(1-\beta)$ in the long run.

Question I has meaning if constant $V(g)$ is assumed; question II if constant $U(W_T)$ is assumed. It seemed to me (and still does) that preferences among probability distributions involving, e.g., 3%, 6% or 9% return in the indefinitely long run are more reasonable to assume than preferences among probability distributions involving a final wealth of, e.g., \$10,000, \$15,000 or \$20,000 in the long run. I will not try to further argue the case for constant $V(g)$ as opposed to constant $U(W_T)$ at this point, other than to encourage the reader to ask himself questions of type I and type II to judge.

In [11] I also argued that even if we were to assume constant $U(W_T)$ rather than constant $V(g)$, we would have to assume that U was bounded (from above and below) in order to avoid paradoxes like those of Bernoulli [3] and Menger [13]. I then show that MEL is asymptotically optional for bounded $U(W_T)$. Merton and Samuelson [12] and Goldman [6] object to my definition of asymptotic optimality, although it is essentially the same as the criteria by which we judge, e.g. a statistic to be asymptotically efficient. Merton and Samuelson proposed, and Goldman adopted, an alternative criterion in terms of the "bribe" required to make a given strategy as good as the optimum strategy. But this bribe criteria seems to me unacceptable, since it violates a basic tenant of game theory—that the normalized form of a game (as described in [17]) is all that is needed for the comparison of strategies. It is not possible to infer the Samuelson-Merton-Goldman bribe from the normalized form of a game. Strategies Ia and Ib in game I may have the same expected utilities, respectively, as strategies IIa and IIb in game II; but a different bribe may be required to make Ia indifferent to Ib than is required to make IIa indifferent to IIb. Strategies IIIa and IIIb in a third game (not necessarily an investment game) may have the same pair of expected utilities as Ia and Ib in game I, or IIa and IIb in game II, but the notion of a bribe may have no meaning whatsoever in game III.[5] Thus unless we are prepared to reject the equivalence between the normalized and extensive form of a game in evaluating strategies, we must reject the Merton-Samuelson-Goldman bribe as part of a precise, formal definition of asymptotic optimality.

5. For example, suppose that strategy (a) has $EU_a=0$ and strategy (b) has $EU_b=\frac{1}{2}$. What bribe will make (a) as good as (b)? Consider the answer, e.g., for one period games I and II in which (a) accepts $W=\frac{1}{2}$ with certainty and (b) elects a 50-50 chance of $W=0$ versus $W=1$. In (I) suppose

$$U=\begin{cases} 0 & \text{for} \quad W \leqslant \frac{1}{2} \\ 10^*(W-\frac{1}{2}) & \text{for} \quad 0.5 \leqslant W \leqslant 0.6 \\ 1 & \quad\quad W \geqslant 0.6 \end{cases}$$

while in II suppose

$$U=\begin{cases} 0 & \text{for} \quad W \leqslant 0.9 \\ 10^*(W-0.9) & \quad\quad 0.9 \leqslant W \leqslant 1.0 \\ 1 & \quad\quad W \geqslant 1.0 \end{cases}$$

In game I, (a) requires a bribe of 0.05; in game II (a) requires 0.45.

IV. Unending Games

Even if we agree that a player playing a fixed finite game should maximize expected utility, we cannot determine whether MEL is asymptotically optimal for a given sequence of games $\{G_i\}$ unless we can agree on criteria for asymptotic optimality. What is needed is either "metacriteria" regarding how to choose criteria of asymptotic optimality, or else an alternate method of analyzing the desirability of strategies for the long run. This section presents such an alternate method, namely the utility analysis of unending games.

Consider a game G_∞ which is like one of the games G_i described above with this one exception: the game G_∞ never terminates. Instead of having a first move, a second move, and so on through a T_ith move, we have an unending sequence of moves. As with a game G_i, a strategy for a G_∞ is a rule specifying the choice of portfolio at each time t as a function of the information available at that time. The only difference is that now the rule is defined for each positive integer $t=1,2,3,\ldots$ rather than only for $1 \leqslant t \leqslant T$.

Given a particular game G_∞ and a strategy (s), a play of the game involves an infinite sequence of "spins of the wheel" and results in an infinite sequence of wealths at each time:

$$(W_0, W_1, W_2, \ldots, W_t, \ldots) \tag{14}$$

where W_0 is initial wealth, and

$$W_t = W_{t-1}^*(1 + \text{return at time } t) \tag{15}$$

as in G_i.

The reader should find it no more unthinkable to imagine an infinite sequence of spins than to imagine drawing a uniformly distributed random variable. For example, if the same wheel is to be spun each time in an unending game, and if this wheel has ten equally probable stopping points, which we may label 0 through 9, then the infinite decimal expansion of a uniform $[0,1]$ random variable may be taken as the infinite sequence of random stopping points of the wheel.[6] If the wheel has sixteen stopping points, then the hexadecimal expansion of the random number may be used. In either case the infinite sequence of wealths $(W_0, W_1, W_2, \ldots)$ is implied by the rules of the game, the player's strategy, and the uniform random number drawn.

In general, a given G_∞ and a given strategy imply a probability distribution of wealth-sequences $(W_0, W_1, W_2, \ldots)$.

Since G_∞ has no "last period", we cannot speak of "final wealth". We can, however, assume that the player has preferences among alternate wealth-sequences: e.g., he may prefer the sequence of passbook entries provided by a savings account which compounds his money at 6%, starting with W_0, to one that compounds it at 3%. Given any two sequences:

$$W^a = (W_0, W_1^a, W_2^a, \ldots)$$

6. The fact that some numbers have two decimal expansions, like 0.4999... versus 0.5000..., may be resolved in any manner without effect on the analysis; since such numbers occur with zero probability.

and

$$W^b = (W_0, W_1^b, W_2^b, \ldots)$$

we may assume that the player either prefers W^a to W^b, or W^b to W^a or is indifferent. Further, we may assume that given a choice between any two probability distributions among sequences of wealth

$$\Pr_A(W_0, W_1, W_2, \ldots)$$

versus

$$\Pr_B(W_0, W_1, W_2, \ldots)$$

he either prefers probability distribution A to B, or B to A, or is indifferent between the two probability distributions.

We shall not only assume that the player has such preferences, but also that he maximizes expected utility. In other words, we assume that he attaches a (finite) number

$$U(W_0, W_1, W_2, \ldots)$$

to each sequence of wealths, and chooses among alternate strategies so as to maximize EU.

The only additional assumption we make about the utility function $U(\ldots)$, is this:

If the sequence $W^a = (W_0, W_1^a, W_2^a, \ldots)$ eventually pulls even with, and then stays even with or ahead of the sequence

$$W^b = (W_0, W_1^b, W_2^b, \ldots)$$

then W^a is at least as good as W^b; i.e., if there exists a T such that

$$W_t^a \geqslant W_t^b \qquad \text{for } t \geqslant T \tag{16}$$

then $U(W^a) \geqslant U(W^b)$. This assumption expresses the basic notion that, in the sense that we have used the terms throughout this controversy, if player A eventually gets and stays ahead of player B (or at least stays even with him) then player A has done at least as well as player B "in the long run".

At first it may seem appropriate to make a stronger assumption that if W_t^a eventually pulls ahead of W_t^b, and stays ahead, then the sequence W^a is preferable to the sequence W^b. In other words, if there is a T such that

$$W_t^a > W_t^b \qquad \text{for } t \geqslant T \tag{16a}$$

then

$$U(W^a) > U(W^b).$$

As shown in the footnote[7], this stronger assumption is too strong in that no utility function $U(W_0, W_1, W_2, \ldots)$ can have this property. Utility functions can however have the weaker requirement in (16).

The analysis of unending games is particularly easy if we assume that the same opportunities are available at each move, and that we only consider strategies which select the same probability distribution of returns each period. We shall make these assumptions at this point. Later we will summarize more general results derived in the appendix to this paper.

Without further loss of generality we will confine our discussion to just two strategies, namely, MEL and any other strategy, and will consider when the expected utility supplied by MEL is at least as great as that supplied by the other strategy. Letting W_t^L and W_t^0 be the wealth at t for a particular play of the game using MEL or the other strategy, respectively,

$$U(W_0, W_1^L, W_2^L, \ldots) \geqslant U(W_0, W_1^0, W_2^0, \ldots) \tag{17}$$

is implied if there is a T such that

$$W_t^L \geqslant W_t^0 \qquad \text{for } t \geqslant T. \tag{18}$$

7. If U orders all sequences $W=(W_0, W_1, W_2, \ldots)$ in a manner consistent with (16a), then in particular it orders sequences of the form

$$\begin{cases} W_0 - \text{given} \\ W_1 - \text{any positive number} \\ W_t = (1+\alpha)\cdot W_{t-1} \qquad \text{for } t \geqslant 2;\ \alpha \geqslant -1. \end{cases} \tag{N.1}$$

Since this family of sequences depends only on W_1 and α, we may here write

$$V(W_1, \alpha) = U(W_0, W_1, W_2, \ldots). \tag{N.2}$$

Then (16a) requires

$$V(W_1^A, \alpha^A) > V(W_1^B, \alpha^B) \qquad \text{if either } \alpha^A > \alpha^B \quad \text{or}$$

$$\alpha^A = \alpha^B \qquad \text{and} \qquad W_1^A > W_1^B. \tag{N.3}$$

For any α let

$$U_{\text{low}}(\alpha) = GLB\ V(W_1, \alpha) \tag{N.4}$$

$$U^{\text{hi}}(\alpha) = LUB\ V(W_1, \alpha).$$

Then (N.3) implies

$$U_{\text{low}}(\alpha) < U^{\text{hi}}(\alpha) \qquad \text{for every } \alpha \tag{N.5a}$$

as well as

$$U_{\text{low}}(\alpha^A) > U^{\text{hi}}(\alpha^B) \qquad \text{if } \alpha^A > \alpha^B. \tag{N.5b}$$

But (N.5b) implies that we can have $U_{\text{low}}(\alpha) < U^{\text{hi}}(\alpha)$ for at most a countable number of values of α, since at most a countable number of values of α can have $U^{\text{hi}}(\alpha) - U_{\text{low}}(\alpha) > 1/N$ for $N = 1, 2, 3, \ldots$. But this contradicts (N.5a).

Equation (2) implies that we have $W_t^L \geqslant W_t^0$ if and only if

$$\frac{1}{t}\sum_{i=1}^{t}\log(1+r_i^L) \geqslant \frac{1}{t}\sum_{i=1}^{t}\log(1+r_i^0). \tag{19}$$

Thus (18) will hold in any play of the game in which there exists a T such that

$$\frac{1}{t}\sum_{i=1}^{t}\log(1+r_i^L) \geqslant \frac{1}{t}\sum_{i=1}^{t}\log(1+r_i^0) \qquad \text{for all } t \geqslant T. \tag{20}$$

Or, if we let

$$y_i = \log(1+r_i) \tag{21}$$

(20) may be written as

$$\frac{1}{t}\sum_{i=1}^{t} y_i^L \geqslant \frac{1}{t}\sum_{i=1}^{t} y_i^0 \qquad \text{for } t \geqslant T. \tag{22}$$

Under the present simplified assumptions

$$Ey^L > Ey^0 \tag{23}$$

by definition of MEL. But for random variables $y_1, y_2, \ldots$ with identical distributions and with (finite) expected value μ, we have

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}(y_i-\mu)=0 \tag{24}$$

except for a set of probability measure zero. In other words

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t} y_i=\mu \tag{25}$$

except for a set of sequences which have (in total) zero probability of occurrence (c.f. the strong law of large numbers in [4] or [5]). But (23) and (25) imply (as a simple corollary of the definition of the limit of sequence) that there exists T such that

$$\frac{1}{t}\sum_{i=1}^{t} y_i^L > \frac{1}{t}\sum_{i=1}^{t} y_i^0 \qquad \text{for } t \geqslant T \tag{26}$$

except on a set of probability zero; hence (17) holds except on a set of measure zero. Since

$$EU=\int U(W_0, W_1, \ldots)\,dP(W_0, W_1, \ldots) \tag{27}$$

is not affected by arbitrarily changing the value of U on a set of measure zero, we have

$$EU(W_0, W_1^L, W_2^L, \ldots) \geqslant EU(W_0, W_1^0, W_2^0, \ldots). \tag{28}$$

Thus, given our simplifying assumption of an unchanging probability distribution of returns for a given strategy, the superiority of MEL follows quite generally.

The case in which opportunities change from period to period and, whether or not opportunities change, strategies may select different distributions at different times, is treated in the appendix. It is shown there that if a certain continuity condition holds, then MEL is optimal quite generally. If this continuity condition does not hold, however, then there can exist games for which MEL is not optimal.

In this respect the results for the unending game are similar to those for the sequence of games with constant $V(g)$. In the latter case we found that MEL was asymptotically optimal for the sequence of games if $V(g)$ was continuous, but could fail to be so if $V(g)$ was discontinuous. In the case of the unending game, the theorem is not concerned with asymptotic optimality in a sequence of games, but optimality for a single game. Given a particular continuity condition, MEL is the optimum strategy.

V. Conclusions

The analysis of investment for the long run in terms of the weak law of large numbers, Breiman's strong law analysis, and the utility analysis of unending games presented here each imply the superiority of MEL under broad assumptions for the hypothetical investor of [1], [2], [7], [8], [9], [10]. The acceptance or rejection of a similar conclusion for the sequence-of-games formalization depends on the definition of asymptotic optimality. For example, if constant $V(g)$ rather than constant $U(W_T)$ is assumed, as this writer believed plausible on *a priori* grounds, then the conclusion of the asymptotic analysis is approximately the same (even in terms of where MEL fails) as those of the unending game.

I conclude, therefore, that a portfolio analyst should not be faulted for warning an investor against choosing E, V efficient portfolios with higher E and V but smaller $E\log(1+R)$, perhaps not even presenting that part of the E, V curve which lies above the point with approximate maximum $E\log(1+R)$, on the grounds that such higher E, V combinations have greater variability in the short run and less "return in the long run".

Appendix

Using the notation of footnote 7, we will show that if $U_{\text{low}}(\alpha) = U^{\text{hi}}(\alpha)$ for all α then MEL is an optimum strategy quite generally; whereas, if $U^{\text{hi}}(\alpha) > U_{\text{low}}(\alpha)$ for some α_0, then a game can be constructed in which MEL is not optimum. $U_{\text{low}}(\alpha) = U^{\text{hi}}(\alpha)$ for all α is the "continuity condition" referred to in the text.

For $v = L$ or 0, indicating the MEL strategy or some other strategy, respectively, we define

$$y_t^v = L_t^v + u_t^v \tag{29}$$

where

$$L_t^v = E\{y_t^v \mid L_1^v, L_2^v, \ldots, L_{t-1}^v, u_1^v, u_2^v, \ldots, u_{t-1}^v\} \tag{30}$$

is the expected value of y_t^v given the events prior to time t. From this follows

$$E\{u_t^v \mid L_1^v, \ldots, u_{t-1}^v\} = 0. \tag{31}$$

The u_t^v (for a given v) are thus what Doob [4] refers to as "orthogonal" random variables, and Feller [5] calls "completely fair" random variables. Therefore, writing var for variance,

$$\sum_{n=1}^{\infty} \frac{\operatorname{var}(u_n^v)}{n^2} < \infty \tag{32}$$

implies

$$\frac{1}{n} \sum_{t=1}^{n} u_t^v \qquad \text{converges to 0 almost always.} \tag{33}$$

(In particular, (32) holds if the $\operatorname{var}(u_n^v)$ are bounded.) In addition to now assuming condition (32) we will also assume that the game is such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} L_i^L \qquad \text{exists almost always.} \tag{34}$$

This is the case, for example, when the same distributions are available each time, whether or not "the other" strategy uses a constant distribution. Since $L_i^L \geqslant L_i^0$ always, we have

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} L_i^L = \limsup_n \frac{1}{n} \sum_{i=1}^{n} L_i^L \geqslant \limsup_n \frac{1}{n} \sum_{i=1}^{n} L_i^0 \qquad \text{always.} \tag{35}$$

Thus when (32) holds we have

$$\lim \frac{1}{n} \sum_{i=1}^{n} y_i^L \geqslant \limsup \frac{1}{n} \sum_{i=1}^{n} y_i^0 \qquad \text{almost always.} \tag{36}$$

In general,

$$\alpha = \limsup \frac{1}{n} \sum y_i^v \qquad \text{implies} \qquad U^{\text{hi}}(\alpha) \geqslant U(W_0, W_1^v, W_2^v, \ldots); \tag{37}$$

(since there always exists another series $y_1^*, y_2^*, \ldots$ such that

$$\alpha = \lim \frac{1}{n} \sum_{i=1}^{n} y_i^*$$

and

$$\frac{1}{n} \sum_{i=1}^{n} y_i^* \geqslant \frac{1}{n} \sum_{i=1}^{n} y_i^v \qquad \text{for all } n;$$

hence

$$U^{\mathrm{hi}}(\alpha) \geqslant U(W_0, W_1^*, W_2^*, \ldots) \geqslant U(U_0, W_1^v, W_2^v, \ldots)).$$

If we now add to the assumptions expressed in equations (32) and (34), the assumption that $U^{\mathrm{hi}}(\alpha) = U_{\mathrm{low}}(\alpha)$ for all α, we get directly from (36) and (37) that

$$EU^L \geqslant EU^0.$$

Conversely, the following is an example in which $U^{\mathrm{hi}}(\alpha_0) > U_{\mathrm{low}}(\alpha_0)$ and which MEL is not optimum: let $W_0 = 1$ and suppose that for some fixed positive α we have $U(1, 0.5, 0.5(1+\alpha), 0.5(1+\alpha)^2, \ldots)$ equals

$$U\big(1, (1+\alpha), (1+\alpha)^2, (1+\alpha)^3, \ldots\big) < U\big(1, 1.5, 1.5(1+\alpha), 1.5(1+\alpha)^2, \ldots\big).$$

With such a U-function it would be better to take a 50-50 chance of $W_1 = 0.5$ or 1.5 followed by $W_t = (1+\alpha)W_{t-1}, t \geqslant 2$, rather than have $W_t = (1+\alpha)\cdot W_{t-1}$ with certainty for $t \geqslant 1, \ldots$.

While the above shows that MEL can fail to be optimal when $U^{\mathrm{hi}}(\alpha) > U_{\mathrm{low}}(\alpha)$ for some α, recall that we can have $U^{\mathrm{hi}}(\alpha) > U_{\mathrm{low}}(\alpha)$ for at most a countable number of values of α. Thus MEL is optimal in a game in which

$$\alpha = \lim_n \frac{1}{n} \sum_{i=1}^{n} L_i^L$$

has a continuous distribution, or in which α has a discrete or mixed distribution but in which none of the points of discontinuity of the cumulative probability distribution of α have $U^{\mathrm{hi}}(\alpha) > U_{\mathrm{low}}(\alpha)$.

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