MANAGEMENT SCIENCE
Vol. 30, No. 12, December 1984
Printed in U.S.A.

DIFFERENT MEASURES OF WIN RATE FOR OPTIMAL PROPORTIONAL BETTING*

PETER A. GRIFFIN
Department of Mathematics and Statistics, California State University, Sacramento, California 95819

It is well known that all betting systems applied to independent, repeated, and identically distributed trials will result in the same expected gain per average unit wagered as that which applies to a single trial. This paper develops the asymptotic and paradoxical manner in which that constant win rate is maintained for optimal proportional betting according to the Kelly criterion. The appropriateness of using this traditional measure of win rate for proportional betting is contrasted with that of various alternatives.
(BETTING SYSTEMS; KELLY CRITERION)

1. Introduction

In the classical formulation of proportional betting, the gambler wagers a fixed fraction, f, of his current capital on a sequence of independent coin tosses. Letting F_n be his fortune after n trials (for convenience $F_0 = 1$), then $F_n = \prod_{i=1}^{n}(1 + fX_i)$, where the X_i are independent and identically distributed: $X_1 = 1$ and -1 with respective probabilities $p > 1/2$ and $1 - p$. The gambler's return on investment, net gain divided by total bet, is represented by

$$R_n = (F_n - 1)/f\sum_{k=0}^{n-1} F_k .$$

When $E\log(1 + fX_1) > 0$ (and $F_n \xrightarrow{\text{a.s.}} +\infty$), Ethier and Tavare (1983) show

$$R_n \xrightarrow{D} R = 1/f\sum_{k=1}^{\infty}(1/F_k), \qquad \text{with} \quad P(0 < R < 1) = 1, \tag{1}$$

$$ER < EX_1, \quad \text{and} \tag{2}$$

$$R_n \mid A_n \xrightarrow{D} R \text{ where } A_n\text{, the conditioning event, consists of all sequences with exactly } [np] \text{ wins in the } n \text{ trials.} \tag{3}$$

The "exponential rate of growth" is defined as $G(f) = E\log(1 + fX_1)$ and satisfies $F_n^{1/n} \xrightarrow{\text{a.s.}} e^{G(f)}$. Since it is natural to imagine F_n increasing due to multiplication rather than addition, $e^{G(f)}$ provides a measure of the rate of increase of capital on a single trial (Thorp 1969). The choice $f = 2p - 1 = EX_1$ (for which $G(f) = f^2/2 + f^4/12 + \cdots + f^{2k}/2k(2k-1) \ldots$ is positive) maximizes G and results in a betting system possessing a variety of optimal properties (Kelly 1956, Breiman 1961, and Finkelstein and Whitley 1981). Ethier and Tavare show further that when the gambler bets this optimal fraction (commonly called using "the Kelly system" or "optimal

*Accepted by William T. Ziemba; received June 1983. This paper has been with the author 4 months for 2 revisions.

0025-1909/84/3012/1540$01.25

proportional betting") and $p \to 1/2^+$ $(f \to 0)$, then

$$R/f = R/EX_1 \xrightarrow{D} \text{exponential (2) with density } g(y) = 2e^{-2y}, \quad \text{and} \tag{4}$$

$$ER/EX_1 \to 1/2. \tag{5}$$

These limiting results generalize to games having payoffs other than $+1$ or -1 and are of particular interest since most realistic examples of proportional betting are not likely to involve large advantages.

At first glance (2) and (5) appear to contradict Epstein's (1977) Theorem I, p. 53: "*If a gambler risks a finite capital over a large number of plays in a game with constant single-trial probability of winning, losing, and tying, then any and all betting schemes lead ultimately to the same value of mathematical expectation of gain per unit amount wagered.*" The confusion results if one fails to distinguish Epstein's interest, namely $E(\text{Win})/E(\text{Bet})$, the average win divided by the average bet, from Ethier and Tavare's, which involves $E(\text{Win/Bet})$, the average of individual win rates. Our purpose is to clarify and discuss these two possible measures of win rate and then to illustrate the rather peculiar manner in which $E(\text{Win})/E(\text{Bet})$ maintains its constant value of $f = EX_1 = 2p - 1$ as time unfolds.

2. $E(\text{Win})/E(\text{Bet})$ and $E(\text{Win/Bet})$, How Do They Differ?

We digress from the topic of proportional betting to consider a gambler who possesses three dollars and has the specific goal of winning one dollar at the unfavorable game of casino craps ($p = 244/495$, Epstein). Dubins and Savage (1965) show that such a gambler's optimal strategy is the familiar "double up if you lose" martingale which will result in three possible sequences of wins(W) and losses(L) as follows:

	Total Win	Total Bet	Win/Bet	Probability
W	1	1	1/1	$p = 0.4929$
LW	1	3	1/3	$(1-p)p = 0.2500$
LL	-3	3	$-3/3$	$(1-p)^2 = 0.2571$

In summary, $E(\text{Win}) = -0.0285$, $E(\text{Bet}) = 2.0141$, $E(\text{Win/Bet}) = 0.3191$, while $E(\text{Win})/E(\text{Bet}) = -0.0141 = EX$, where X is the result of a unit bet at craps. This example not only illustrates Epstein's Theorem I, but also shows that $E(\text{Win/Bet})$ need not even have the same algebraic sign as EX in some instances.

Indeed if we imagine thousands of such gamblers each initially capitalized with three dollars and desiring to win one dollar at an obliging casino offering craps, we can articulate the distinction between the two measures of win rate. We ask each of the gamblers, after they exit from the casino, to write down on a piece of paper their individual win rates, either 1/1, 1/3, or $-3/3$. When we average the fractions on these pieces of paper, the result will approximate $E(\text{Win/Bet}) = 0.3191$. On the other hand, if we asked the silver dollars which were bet what happened to them, their collective response would be "We lost, and at about the expected rate." This would be reflected by the sum of the numerators on the pieces of paper divided by the sum of the denominators approximating $EX = -0.0141$.

Thus $E(\text{Win/Bet})$ quantifies how, on average, individuals perceive their win rates. $E(\text{Win})/E(\text{Bet})$ shows us in what direction and how fast the money is flowing.

One of the appeals of $E(\text{Win})/E(\text{Bet})$ as a measure of win rate is that it provides an almost sure limit for the return on investment for either an individual or group of individuals who repeatedly apply any betting scheme to necessarily terminating and independent sequences of trials for which $E(\text{Bet})$ is finite. However, returning to our main interest of the Kelly system, we observe that a proportional bettor is hypothesized to gamble forever without starting over, and thus does not repeat betting sequences which come to an end. This suggests that $E(\text{Win})/E(\text{Bet})$ may not be an appropriate measure for the *individual* gambler in this case.

The temptation to resort to $E(R)$ as an alternative measure must be tempered, as the individual R_n sample paths do not converge at all:

$$P(\lim R_n \text{ exists}) = 0. \tag{6}$$

To prove this, suppose, as occurs infinitely often, the nth trial results in a loss and $F_{n-1} > 1$. Then (6) follows from

$$R_n/R_{n-1} = ((1-f)F_{n-1} - 1)\sum_{k=0}^{n-2} F_k/(F_{n-1} - 1)\sum_{k=0}^{n-1} F_k < 1 - f,$$

since (1) excludes zero as a possible limit. Although $E(R)$ will be the average of a solitary gambler's varying win rates in the long run, he in no sense maintains or approaches this value. What happens is that amounts gained and wagered within fixed, albeit lengthy, periods in the future continue to overwhelm the totals of his entire previous history. Subsequent values of R_n, if spaced sufficiently far apart, behave like almost independent observations from the distribution of R.

The preceding remarks suggest an efficient method of simulating the behavior of R. Instead of using mn random numbers to produce only m observations of R_n and trusting n to be sufficiently large, record all of the consecutive win rates of just one gambler. Provided that observation is begun after F_n has grown enough to dominate completely the -1 in the numerator of R_n, the Markovian nature of $R_n F_n/(F_n - 1)$ (which has the same limiting distribution as R_n) will assert itself and produce a time-homogeneous Markov process governed by the transition function

$$P(x, C) = p 1_C((1+f)x/(1+fx)) + (1-p)1_C((1-f)x/(1+fx)),$$

where 1_C is the indicator function for the set C.

The shapes displayed in Figure 1 were obtained from a million observations each for three cases:

A: $p = 0.7, f = 0.4$, with estimates of $ER = 0.22$ and $ER^2 = 0.09$;
B: $p = 0.8, f = 0.6$, with estimates of $ER = 0.38$ and $ER^2 = 0.24$; and
C: $p = 0.9, f = 0.8$, with estimates of $ER = 0.60$ and $ER^2 = 0.50$.

Surprisingly, the exponential distribution of (4) fits the $p = 0.7$ data reasonably well. But when p is larger we begin to see a pronounced tendency for win rates to cluster near 1, only to be jerked back to about $(1-f)/(1+f)$ by the occurrence of a loss, and then to begin migrating back toward 1 again.

Any interpretation of $E(\text{Win})/E(\text{Bet})$ for proportional betting requires the context of the collective win rate of many individual gamblers, each embarked on separate, independent, optimal odysseys, but playing the same game (same p, $f = 2p - 1$). It is not difficult to confirm the invariance of $E(\text{Win})/E(\text{Bet})$ after any finite number of trials. Independence of the X_i gives $E(F_n) = (1 + f^2)^n$, from which follow $E(f\sum_{k=0}^{n-1} F_k) = ((1+f^2)^n - 1)/f$ and consequently

$$E(\text{Win})/E(\text{Bet}) = E(F_n - 1)/E\left(f\sum_{k=0}^{n-1} F_k\right) = f = EX_1. \tag{7}$$

Wong (1981), who first raised the question of how proportional betting affects win rate, prophesied that a disproportionate contribution to this constant win rate would

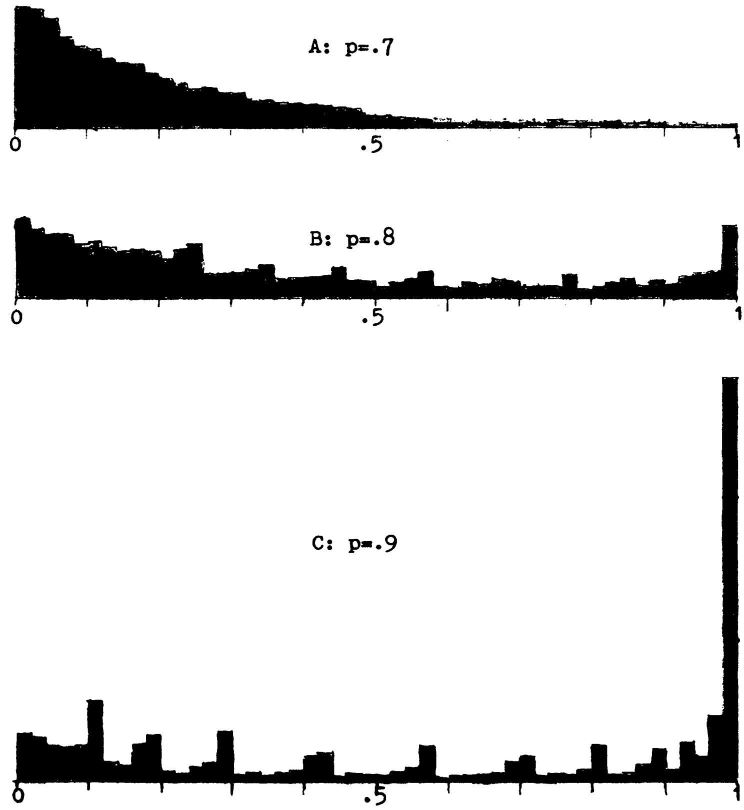

FIGURE 1. Sample histograms illustrating the shape of Ethier's R for selected values of p and optimal betting fractions $f = 2p - 1$. Possible values of R occur on the horizontal scale in intervals of width 0.02.

come from the sequences consisting of an inordinately large number of wins. That this is the case we shall now see.

3. Conditioning $E(\text{Win})/E(\text{Bet})$ on a Fixed Sample Proportion

Let $E_{n,p'}$ stand for conditional expectation given the event of precisely np' wins in n trials. The conditional expected amounts won and bet are

$$W_{n,p'} = E_{n,p'}(F_n - 1) = (1+f)^{np'}(1-f)^{n-np'} - 1 \quad \text{and}$$

$$B_{n,p'} = E_{n,p'}\left(f \sum_{k=0}^{n-1} F_k\right), \quad \text{respectively.}$$

If p_0 is the solution to $(1+f)^{p_0}(1-f)^{1-p_0} = 1$, it also satisfies $W_{n,p_0} = 0$ and has the interpretation of being the sample proportion which results in no change in fortune at all for the proportional bettor. (If $p \sim 1/2$, $p_0 \sim p/2 + 1/4$.) The main result is now established that, when $p' \geqslant p_0$,

$$\limsup W_{n,p'}/B_{n,p'} \leqslant Q(p,p') = \max\left(\frac{p'-p}{p'+p-2pp'}, 0\right). \tag{8}$$

(Values of n are restricted by the requirement that np' be integral.)

The method of proof relies upon the observation that the amount of money won by proportional bettors winning np' games in n trials, $F_n - 1$, is necessarily constant and equal to $W_{n,p'}$ regardless of which of the sequences of results we examine. Only the amount of money bet is variable, depending upon the order in which the wins and losses occur. Hence, when $p' > p_0$,

$$W_{n,p'}/B_{n,p'} = 1/E_{n,p'}\left(f\sum_{k=0}^{n-1} F_k/(F_n - 1)\right). \tag{9}$$

Now, we temporarily change perspective and assume p' as the true probability of success, in which case the fraction of money bet, $f = 2p - 1$, may not be the optimal fraction for p'. We regard the expectation on the right-hand side of (9) as $E(1/R_n \mid A_n)$, where A_n is the event that there are np' wins in the n trials. Since $R_n \mid A_n$ is nonnegative and, by (3), converges in distribution to R, $1/R_n \mid A_n$ converges to $1/R$.

By Billingsley (1968, (5.3)),

$$\liminf E(1/R_n \mid A_n) \geqslant E(1/R). \tag{10}$$

Evaluating the right-hand side of (10),

$$E(1/R) = f\sum_{k=1}^{\infty} E(1/F_k) = f\sum_{k=1}^{\infty} E\left(\prod_{i=1}^{k}(1/(1+fX_i))\right) = f\sum_{k=1}^{\infty}(E(1/(1+fX_1)))^k,$$

$$E(1/(1+fX_1)) = p'/(1+f) + (1-p')/(1-f) = (1 - f(2p'-1))/(1-f^2) < 1$$

iff $p' > p$, and therefore

$$E(1/R) = +\infty \qquad \text{if } p' \leqslant p,$$

$$= \frac{p' + p - 2pp'}{p' - p} \qquad \text{if } p' > p.$$

Thus (8) is established.

Note that

$$\lim W_{n,p'}/B_{n,p'} = 0 \qquad \text{for } p_0 \leqslant p' \leqslant p. \tag{11}$$

This result for $p' = p$ is, by itself, somewhat surprising. Loosely speaking its meaning is that the aggregate amount of money won divided by the total amount wagered approaches zero for all optimal proportional bettors who coincidentally experience the ideal, or expected, proportion of wins and losses in their lengthening sequence of plays. At the same time, each individual in this collection is having his fortune grow unlimitedly at precisely the exponential growth rate desired.

The following numerical calculations, to illustrate (8) when $p = 0.6$, $f = 0.2$, and $p_0 = 0.5503$, suggest that lim sup might be replaced by ordinary limit and inequality with equality for $p' > p$ as well. Also, there may exist an interval of values, $p' < p_0$, for which $W_{n,p'} < 0$ but $\lim W_{n,p'}/B_{n,p'} = 0$.

Values of $W_{n,p'}/B_{n,p'}$

p' \ n:	5	10	100	500	1000	$Q(p, p')$
0.50		− 0.0973	− 0.0724	− 0.0357	− 0.0252	
0.55			− 0.0005	− 0.0001	− 0.00001	
0.59			0.0580	0.0241	0.0141	0.0000
0.60	0.0997	0.0980	0.0734	0.0361	0.0256	0.0000
0.61			0.0879	0.0503	0.0401	0.0209
0.70		0.3028	0.2528	0.2262	0.2219	0.2174
0.80	0.5253	0.5194	0.4724	0.4584	0.4565	0.4545
0.90		0.7508	0.7220	0.7159	0.7150	0.7143

4. An Asymptotically Certain Sequence of Events with Arbitrarily Small Average Win per Unit Bet

An even more surprising result than (11) holds, but first we require a proof of the seemingly obvious fact that

$$W_{n,p'}/B_{n,p'} < W_{n,p''}/B_{n,p''} \quad \text{if} \quad p' < p''. \tag{12}$$

We prove this by supposing $n = v + d$ games played by a proportional bettor have resulted in v victories and d defeats with some given order specified for the results. Let a particular trial which results in a loss be identified, with B being the amount of money wagered up to and including this loss and A the amount wagered after the loss. For this sequence the total amount won is $(1+f)^v(1-f)^d - 1$, while the total amount bet is $B + A$. Now, change the specified loss to a win so the resultant sequence of $v+1$ wins and $d-1$ losses is associated with a gain of $(1+f)^{v+1}(1-f)^{d-1} - 1$ and a total wager of $B + (1+f)A/(1-f)$. It is simple to show that the win rate in the latter case is larger than in the former and hence that

$$\frac{B + (1+f)A/(1-f)}{(1+f)^{v+1}(1-f)^{d-1} - 1} < \frac{B + A}{(1+f)^v(1-f)^d - 1}. \tag{13}$$

Suppose we now change each of the d losses in all $\binom{n}{d}$ sequences of v wins and d losses into a win. We will thus have created all $\binom{n}{d-1}$ sequences of $v+1$ wins and $d-1$ losses, however with a multiplicity of $v+1$. Summing all $(v+1)\binom{n}{d-1}$ possible values of the left-hand side and $d\binom{n}{d}$ values of the right-hand side of (13), we obtain

$$(v+1)\binom{n}{d-1}\frac{B_{n,p''}}{W_{n,p''}} < d\binom{n}{d}\frac{B_{n,p'}}{W_{n,p'}}, \qquad \text{where} \quad p'' = (v+1)/n \quad \text{and} \quad p' = v/n.$$

Canceling $(v+1)\binom{n}{d-1} = d\binom{n}{d}$ and inverting yields (12).

Now, let $\epsilon > 0$ be given and define

$$p'' = \frac{1 + f + f\epsilon + \epsilon}{2(1 + f\epsilon)} > p = (1+f)/2,$$

for which $Q(p, p'') = \epsilon$. Let A_n be the event that the sample proportion of wins in n trials is less than p''. The quotient $E(\text{Win} \mid A_n)/E(\text{Bet} \mid A_n)$ is a weighted average of the values of $W_{n,p'}/B_{n,p'}$ with $p' < p''$, and therefore, using (12),

$$\limsup E(\text{Win} \mid A_n)/E(\text{Bet} \mid A_n) \leqslant \limsup W_{n,p''}/B_{n,p''} \leqslant \epsilon, \tag{14}$$

with $P(A_n) \to 1$.

The unconditional value of $E(\text{Win})/E(\text{Bet})$ after n trials is also a weighted average of the $W_{n,p'}/B_{n,p'}$. (Weighting is by the relative contribution to $E(\text{Bet})$ for each possible p'.) Thus its constant value of $f = 2p - 1$ results almost entirely from the increasingly improbable sample proportions distinctly greater than p itself.

5. Long-Run Fantasy

Suppose a host of gambling angels wager the fraction $f = 2p - 1$ of their fortunes on independent sequences of tosses of a biased coin, $p > 1/2$. We look in on them after, for instance, large n trials each and ask all of them to write their individual fractions of total win divided by total bet on separate pieces of paper. Naturally almost all of them are ahead of their gambles and most have won huge sums.

Equation (7) suggests that their aggregate win divided by their enormous collective wager (the sum of the numerators divided by the sum of the denominators) will be nearly $EX_1 = f$ and will, by (2), exceed the average value of all the fractions on their pieces of paper. If the game is not too favorable (p close to $1/2$), (4) and (5) predict that the shape of a histogram of the values of all the individual win rates will resemble an exponential distribution and that the unweighted average of all the gamblers' win rates will be about half of the collective win rate (weighting by amount bet).

If we isolate those who have coincidentally won precisely np games, thus experiencing the most probable result, (3) guarantees the distribution of this group's individual win rates will appear much the same as that of the entire group. Nevertheless, by (11), these angels, who in a sense have nothing to complain about, will have a collective win rate near zero.

When we look in on the entire host again after another large n trials, all of the previous remarks apply. But, by (6), almost none of the gamblers will have the same win rates we observed on our first inspection; it's as if a divine wind redistributed the fractional win rates they all originally displayed.

A gambler who recorded his own up to date win rate every thousand trials or so would discover, in reminiscence, that this collection of figures also would have a distribution like that of the entire group. But for the individual desiring a constant, predictable return on his investment there would be no hope.

Perhaps the most astonishing aspect of this asymptotic behavior is that no matter how favorable the game, so long as $p < 1$, (14) would apply. Thus even if they play a game with $p = 0.99$, bet $f = 0.98$ of their current capital at each trial, and get astronomically rich very fast, the proportion of these gamblers having long-run sample proportions of success less than $p'' = 0.9902$, and hence collective win rate unlikely to exceed $\epsilon = 0.01$, would approach 1.

6. Remarks

Many numerical calculations and simulations not presented here strengthen the conjectures near the end of §3 and also suggest the following:

(a) The absolute value of $W_{n,p'}/B_{n,p'}$ is decreasing in n for all p' and, as we might anticipate, $\lim W_{n,p'}/B_{n,p'} = 2p' - 1$ if $p' \leqslant 1/2$.

(b) $E_{n,p}(R_n)$ increases in n and ER_n decreases in n, both with the same limit of ER.

(c) If R_{nm} is the collective win rate of m optimal proportional bettors each independently playing the same game n times, it converges in distribution to R regardless of what fixed value is assigned to m. This situation is apparently a different kettle of fish entirely from fixing n, however large, and then letting m increase without bound, in which case the convergence is to the constant f.

(d) The same qualitative conclusions about $E(\text{Win})/E(\text{Bet})$ apply to games with payoffs other than the two values of $+1$ and -1 (for which the optimal fraction f is usually not equal to EX_1). The formulation would be more awkward and the proof, no doubt, more intricate.

In practice, the asymptotic curios of this paper are unlikely to be realized to a perceptible degree, this for a couple of reasons. The number of trials and gamblers necessary would require both an amount of money far in excess of the wealth of this planet and time to find gambling opportunities beyond the human lifespan. Also, the unspecified magnitude of the "host" in §5 is necessitated because the nature of the distribution of R_{nm} is unknown: with large n fixed, the value of m necessary for the collective win rate to stabilize near $E(\text{Win})/E(\text{Bet})$ could be very large indeed.

The results do serve, however, to remind us that the rationale for proportional betting involves neither the expectation of wealth itself nor conventional measures of win rate, but rather such factors as logarithmic utility, long-term capital growth, and probability of ruin. Some interesting examples of the extent to which ordinary expectation is sacrificed by proportional betting can be found in Finkelstein and Whitley.[1]

[1] This paper would not have been possible without Stewart Ethier's invaluable correspondence.

References

BILLINGSLEY, PATRICK, *Convergence of Probability Measures*, Wiley, New York, 1968.

BREIMAN, LEO, "Optimal Gambling Systems for Fair Games," *Proc. 4th Berkeley Sympos. on Math. Stat. and Prob. 1* (1961), 65–78.

DUBINS, LESTER AND LEONARD SAVAGE, *How to Gamble If You Must*, McGraw-Hill, New York, 1965.

EPSTEIN, RICHARD, *The Theory of Gambling and Statistical Logic*, rev., Academic Press, New York, 1977.

ETHIER, S. AND S. TAVARE, "The Proportional Bettor's Return on Investment," *J. Appl. Probab.*, 20 (1983), 563–573.

FINKELSTEIN, MARK AND ROBERT WHITLEY, "Optimal Strategies for Repeated Games," *Adv. in Appl. Probab.*, 13 (1981), 415–428.

KELLY, J. L., JR., "A New Interpretation of Information Rate," *Bell System Tech. J.*, 35 (1956), 917–926.

THORP, E. O., "Optimal Gambling Systems for Favorable Games," *Rev. Internat. Statist. Inst.*, 37, 3 (1969), 273–293.

WONG, STANFORD, "What Proportional Betting Does to Your Win Rate," *Blackjack World*, 3 (1981), 162–168.