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THE TREYNOR CAPITAL ASSET PRICING MODEL

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History generally accords the development of the single-period, discrete-time Capital Asset Pricing Model (CAPM) to the works of Sharpe (1964), Lintner (1965a,b) and Mossin (1966). We explore the early work of another notable financial economist, Jack L. Treynor, who also deserves credit for the original Capital Asset Pricing Model because of his revolutionary manuscripts—"Market Value, Time, and Risk", Treynor (1961), and "Toward a Theory of Market Value of Risky Assets", Treynor (1962)—which were circulated during the 1960s in mimeographed draft form but have never been published in an academic or practitioner journal. Mr. Treynor's early work appears to have predated and anticipated Sharpe (1964), Lintner (1965a,b) and Mossin (1966). However, while financial economists initially credited Mr. Treynor for his innovation, the Treynor CAPM has not enjoyed a broad public reach. This, apparently, is the reason Mr. Treynor is not consistently recognized as one of the primary architects of the CAPM.

1 Introduction

In 1981 Fischer Black wrote an open letter to Jack Treynor, whose 13-year tenure as the editor of the *Financial Analysts Journal* was then coming to a close; in his letter, Dr. Black stated, "You developed the capital asset pricing model before anyone else."[1] The present paper investigates this assertion and concludes that, like so many of Fischer Black's other beliefs, it seems to be accurate.

Popular history generally accords the initial development of the Capital Asset Pricing Model (CAPM) to the works of Sharpe (1964), Lintner (1965a,b), and Mossin (1966).[2] After a decade of academic attempts, the most frequently cited likely being Black *et al*. (1972) and Fama and MacBeth (1973), to substantiate or refute the validity of the CAPM as a positive economic model, Roll (1977) demonstrated that, since the "market portfolio" specified by the model is immeasurable, the CAPM can never be empirically tested conclusively. Nevertheless, the CAPM continues to inspire theoretical and

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empirical research. As Dr. Black recognized, Jack Treynor also deserves credit for the original CAPM because of his revolutionary manuscripts, "Market Value, Time, and Risk" and "Toward a Theory of Market Value of Risky Assets," which were circulated during the 1960s in mimeographed draft form but have never been published in an academic or practitioner journal.[3]

The reference dates cited in the literature usually refer to Mr. Treynor's CAPM as Treynor (1961)[4] and Dr. Sharpe's CAPM as Sharpe (1964). However, it would be a mistake to date the former work with the date it is generally credited with being written and the latter paper with its date of publication. Mr. Treynor and Dr. Sharpe developed their original models independently and almost concurrently. It is well known that by the end of 1961, Dr. Sharpe had extended the final chapter of the doctoral dissertation he had begun in 1960 into a paper that he first presented in January 1962 to the Quadrangle Club in Chicago. He submitted this paper to the *Journal of Finance* in 1962, and it was rejected. Upon resubmission, it was published as Sharpe (1964).

In 1958, Jack Treynor was employed by Arthur D. Little. That summer he took a three-week vacation to Evergreen, Colorado, during which he produced forty-four pages of mathematical notes on capital asset pricing and capital budgeting. Over the next two years, Mr. Treynor refined his notes into what is in all likelihood the first CAPM. Mr. Treynor gave a copy of this early model to John Lintner at Harvard in 1960. While in business school at Harvard from 1953 through 1955, Mr. Treynor had taken nearly every finance course offered, and though he signed up for Dr. Lintner's economics course he was forced to cancel due to a schedule conflict. In 1960, Dr. Lintner was the only economist he knew even slightly.

Mr. Treynor refined his 1960 model into the 45-page "Market Value, Time, and Risk" (Treynor, 1961). Treynor (1961) developed the CAPM using the concept of experiment space to quantify risk and risk relations.[5] Without his knowledge or encouragement, one of Mr. Treynor's colleagues sent the draft to Merton Miller in 1961, after Dr. Miller had moved to the University of Chicago from Carnegie Institute of Technology. Dr. Miller sent the paper to Franco Modigliani at MIT in the spring of 1962, and Dr. Modigliani invited Mr. Treynor to embark on a program of graduate work at MIT under his supervision. Mr. Treynor did so during the 1962–1963 academic year; in addition to Dr. Modigliani's course, he took Bob Bishop's price theory and Ed Kuh's econometrics courses, among others.

By the fall of 1962, Mr. Treynor had consolidated the first part of Treynor (1961), on the single-period model, into "Toward a Theory of Market Value of Risky Assets," and presented it to the MIT finance faculty. Although this famous paper has generally been cited in the literature as "Treynor (1961)", it was written as an independent piece in 1962, and we refer to it herein as Treynor (1962). Treynor (1962) uses the first-order conditions for exposition, which provides greater clarity than the experiment space method employed in its parent. By the spring of 1963, Mr. Treynor had consolidated the second part of Treynor (1961), and presented it to the MIT faculty. This third paper, "Implications for the Theory of Finance," Treynor (1963), appears to have been the first development of an intertemporal, multiperiod CAPM. It was later radically rethought and rewritten by Fischer Black, and appeared as Treynor and Black (1976). In the summer of 1963, Mr. Treynor returned to Arthur D. Little and began to work on applications of his theory to the problem of portfolio analysis.[6] Mr. Treynor subsequently published more than fifty papers in the *Harvard Business Review*, *Financial Analysts Journal*, *Journal of Portfolio Management*, *Journal of Finance*, *Journal of Business* and the *Journal of Accounting Research*, including Treynor (1965)—on measuring selection—and Treynor and Mazuy (1966)—on measuring timing—as well as Treynor and Black (1973)—on the appraisal problem referred to in Treynor (1962).[7]

Thus it appears that, while both models were developed nearly simultaneously, the conception and initial drafting of Mr. Treynor's CAPM pre-dated that of Dr. Sharpe's. Treynor (1961, 1962) was arguably the first CAPM to derive the linear relationship between expected return and covariance with the market portfolio and also to conclude that in equilibrium, the market itself is the single optimal mean–variance efficient portfolio.

In spite of its lack of publication, Treynor (1962) has received some credit; it is possibly the most frequently cited unpublished work in the financial economics literature. Treynor (1962) is one of the very few unpublished papers listed [entry 5419] in the authoritative financial research bibliography of Brealey and Edwards (1991).

Several preeminent financial economists, including Sharpe (1964), have cited Treynor (1962). Jensen (1972b) describes two primary lines of inquiry into positive applications of the normative Markowitz portfolio selection framework: "(1) Tobin's (1958) work utilizing the foundations of portfolio theory to draw implications regarding the demand for cash balances and (2) the general equilibrium models of asset prices derived by Treynor (1961[*sic*]), Sharpe (1964), Lintner (1965a,b), Mossin (1966), and Fama (1968)."[8] Jensen (1972b) further describes empirical tests of the CAPM using evidence from mutual fund returns, and asserts that the first ever portfolio evaluation model was also a Treynor model—Dr. Jensen lists this line of research in chronological order, as Treynor (1965), Sharpe (1966), and Jensen (1968, 1969). It seems that Dr. Jensen viewed Mr. Treynor as the pioneer of both the theoretical development and practical use of the CAPM. Black, Jensen, and Scholes, in their famous 1972 empirical tests, state in their introduction that "… the best known [general equilibrium model of the pricing of capital assets] is the mean–variance formulation originally developed by Sharpe (1964) and Treynor (1961[*sic*]), and extended and clarified by Lintner (1965a,b), Mossin (1966), Fama (1968), and Long (1972)."[9] Fischer Black, in his "other" paper of 1972, states, "The first writers to deal adequately with uncertainty are Sharpe (1964), Treynor (1961[*sic*]), Lintner (1965[a]), Mossin (1966), and Fama (1968)."[10] Clearly Dr. Black viewed Mr. Treynor as having developed one of the first capital asset pricing models; in fact, Black (1981) indicates plainly that he believed Mr. Treynor was the first ever to develop the CAPM as we understand it today.

Two of the most important works in financial economics cite Treynor (1962). Black and Scholes (1973) introduce their elegant options pricing model (which has been called "the most successful theory not only in finance, but in all of economics" by Professor Ross[11]) by informing us "The inspiration for this work was provided by Jack L. Treynor" [in his unpublished memorandums, "Implications for the Theory of Finance" and "Toward a Theory of Market Value of Risky Assets"]. Black and Scholes (1973) provide a derivation of their differential equation using the CAPM, which they attribute to Treynor (1962), Sharpe (1964), Lintner (1965a) and Mossin (1966). Ross (1976, 1977), in his magnificent Arbitrage Pricing Theory, also references Treynor (1962).

2 Mr. Treynor's development of the CAPM

The published version of Treynor (1962), in Korajczyk (1999), is nearly identical to the original 1962 mimeo. Edits consist primarily of minor typographical corrections. Note that Mr. Treynor's notation is economical: Double summations, typically denoted with two summation signs, e.g., $\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij}$, which is the customary representation of a sum of sums $\sum_{i=1}^{N}(\sum_{j=1}^{N} X_i X_j \sigma_{ij})$, sometimes expressed less formally as $\sum\sum X_i X_j \sigma_{ij}$, when the nature of the summation is clear from context, are denoted in Treynor (1962) as follows:[12] $\sum_{ij} X_i X_j \sigma_{ij}$, and occasionally simply as $\sum X_i X_j \sigma_{ij}$.

In this section we review the development of Treynor (1962). We note the mathematical equivalence of Treynor's risk premium measure a_i for capital assets, and that of its linear relationship in equilibrium with the single period expected return, to those of Sharpe (1964) and Lintner (1965a,b). The reinterpretation of Sharpe (1964) provided by Fama (1968) asserted the equivalence of the latter two models, while the proofs given in Stone (1970) formally established the equivalence of the Sharpe–Fama, Lintner, and Mossin models. It seems clear that the model of Treynor (1962) is equivalent to the three later models. Treynor (1962) can certainly be considered to reside neatly within the two-parameter functional representation of Stone (1970), as a special case of Stone's general model, alongside each of the later models.

The introduction of Treynor (1962) (the first two paragraphs of p. 15)[13] lists the aims of this "highly idealized" capital market model as being: (1) to demonstrate that optimal behavior of the agents leads to Proposition I of Modigliani and Miller (1958); (2) to investigate the relation between risk and investment value; and (3) to distinguish between insurable and uninsurable risk. Mr. Treynor approached capital asset pricing from the perspective of corporate cost-of-capital decision-making. While still in business school, Mr. Treynor "resolved to try to understand the relation between risk and the discount rate [for making long-term plant investment decisions.]"[14] This explains the focus of Treynor (1962) on Proposition I of Modigliani and Miller (1958), which asserts that, in equilibrium, "the market value of any firm is independent of its capital structure and is given by capitalizing its expected return at the rate ρ_k appropriate to its class."[15]

Treynor (1962), at the bottom of p. 15 and all of p. 16, discusses the seven primary assumptions of Treynor's market model: (1) no taxes; (2) no market frictions; (3) trading does not affect prices; (4) agents maximize utility in the sense of Markowitz; (5) agents are risk-averse; (6) a perfect lending market exists; and (7) agents have identical knowledge of the market and agree in their forecasts of future values. These assumptions are listed in Table 1 of the present paper, where they are compared with those of Sharpe (1964), Lintner (1965a), and Mossin (1966).

Mossin (1966) notes that the assumption of identical perceptions among agents about the probability distributions of the yields of risky assets is not crucial, and also that specification of quadratic utility functions (through the assumption of agents' focus on the first two moments of the probability distribution, which he does invoke, along with acceptance of the von-Neumann–Morgenstern utility axioms) is unnecessary.[16] While Dr. Sharpe explicitly allows the covariance matrix of the risky assets to be singular, Dr. Lintner and Dr. Mossin explicitly require it to be positive definite and therefore non-singular—Lintner (1965a), p. 21, and Mossin (1966), p. 771—and Treynor (1962) implicitly requires non-singularity.

Treynor (1962) develops the CAPM in the last paragraph on p. 16 through p. 20. The exposition begins by decomposing expected return into (1) a risk-free component and (2) a risk-premium component. By definition, Treynor's risk-free component is equivalent to that of Lintner: Dr. Lintner defines the risk-free component r^* as "the *interest rate* on riskless assets or borrowing" (Lintner, 1965a, p. 16), while Mr. Treynor defines the risk-free component r

Table 1 Assumptions of the models.

Assumption	Treynor (1962)	Sharpe (1964)	Linter (1965)	Mossin (1966)
No taxes	Explicit	Implicit	Explicit	Implicit
No frictions (transactions costs)	Explicit	Implicit	Explicit	Implicit
Agents are price takers who all face identical prices	Explicit	Implicit	Explicit	Implicit
Agents maximize expected utility of future wealth	Explicit	Explicit	Explicit	Explicit
Utility represented as a function of return and risk	Explicit	Explicit	Explicit	Explicit
All agents agree that variance (or standard deviation) is the measure of security risk	Explicit	Explicit	Explicit	Explicit
Agents prefer more return to less and display risk aversion	Explicit	Explicit	Explicit	Explicit
A riskless asset (paying an exogenously determined positive rate of interest) exists, and all investors agree that it is riskless	Explicit	Explicit	Explicit	Explicit
All agents share the same subjective probability distribution of expected future prices	Explicit	Explicit	Explicit	Explicit
Fractional shares may be held	Implicit	Implicit	Explicit	Explicit
Short sales are allowed	Explicitly allowed	Explicitly disallowed	Explicitly allowed	Explicitly allowed
Leverage is allowed	Explicitly allowed	Explicitly disallowed	Explicitly allowed	Implicity allowed
The number of shares of each security is constant	Implicit	Implicit	Implicit	Implicit
Agents share the same single period time horizon	Explicit	Explicit	Implicit	Implicit

as the [perfect] "lending rate", which is a component of his one-period discount factor b. Since Treynor (1962), p. 17, defines $b = 1/(1 + r)$, this is equivalent to defining r as the growth factor. Therefore, expected performance is given by rC [the return on capital at the risk-free rate] plus $(1 + r) \sum x_i a_i$ [the expected return due exclusively to the risk premia]. Likewise, by definition, Treynor's risk premium, a_i, is equivalent to Lintner's risk premium $\bar{x}_i$.[17]

Treynor (1962) defines the expected portfolio risk premium μ as the present value of the portfolio risk premium, and he derives the linear relation between risk and expected return on pp. 18 and 19. Mr. Treynor defines the covariance matrix using Dr. Markowitz' formula.[18] He also defines portfolio variance using the Markowitz formulation.[19]

Next, as in Tobin (1958), p. 83, Mr. Treynor sets out to find the linear relation: first, Treynor defines expected performance in terms of Tobin's "non-negative scalar k," and proceeds to minimize portfolio variance subject to expected performance. He forms the Lagrangian and inverts the covariance matrix, arriving after some substitution and algebra at the reward per risk equality $\mu^2/\sigma^2 = 2k/\lambda$, and given the definition of λ,[20] demonstrates that k is a linear function of σ. This is so because the reward per risk ratio μ^2/σ^2 is equivalent to the weighted sum of sums on the covariance matrix inverse, the equality shown as $\mu^2/\sigma^2 = \sum_{ij} a_j B_{ji} a_i$ on p. 19.

This is equivalent to Eq. (3.25), the linear opportunity locus, on p. 84 of Tobin (1958): Squaring both sides of Tobin's (3.25),[21] we obtain

$$\mu_R^2 = \sigma_R^2 \sum_i \sum_j r_i r_j V_{ij},$$

which (since Tobin sets $[V_{ij}] = [V_{ij}]^{-1}$, and therefore V_{ij} in expression (3.25) is the analogue of

Treynor's B_{ji}, the inverse of the covariance matrix, and also since Tobin's r_i corresponds to Treynor's a_i) is equivalent to

$$\mu^2 = \sigma^2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_j B_{ji} a_i.$$

Since the expression

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_j B_{ji} a_i$$

is expressed by Treynor as

$$\sum_{ij} a_j B_{ji} a_i,$$

and this is shown to be equal to μ^2/σ^2 in Treynor's equation, it is clear that Mr. Treynor's formula equals Dr. Tobin's (3.25). Likewise, Mr. Treynor's discussion at the bottom of p. 19 is analogous to that given by Dr. Tobin at the bottom of p. 83 and the top of p. 84. Mr. Treynor's development to this point is, therefore, indeed equivalent to that of Tobin (1958).

Dr. Lintner, too, presents proofs of Tobin's separation theorem, following Fisher (1930). Dr. Lintner refers to an environment in which agents have identical probability beliefs, or homogeneous expectations, as "idealized uncertainty," and it is under these conditions that Dr. Lintner arrives at the following conclusions: In equilibrium, (1) the same combination of risky assets will be optimal for every investor, (2) the investment amounts invested in each risky asset will be equivalent to the ratio of the aggregate market value of the ith risky asset to the total aggregate value of all risky assets, and (3) each investment amount in the individual risky assets must therefore be a positive amount.[22] Mr. Treynor reaches the first conclusion in his discussion at the bottom of p. 19 ("The holdings of any two investors are thus identical, up to a factor of proportionality,") and the latter two conclusions in his discussion at the top of p. 21 ("ideally the investor will hold shares in each equity in proportion to the total number of shares in the market—and the latter share quantities are always positive.")

3 Comparison of models

The CAPM was built upon the single-period discrete-time foundation of Markowitz (1952, 1959) and Tobin (1958). Although Dr. Sharpe himself did not conclude in Sharpe (1964) that the market itself is the single optimal portfolio, Fama (1968) offered an interpretation that did, and also reconciled the Sharpe and Lintner models. Lintner (1965a) reached exactly the same conclusions as Treynor (1961, 1962). Mossin (1966) clarified Sharpe (1964) by providing a more precise specification of the equilibrium conditions.

Each of the models makes generally similar assumptions. A summary of the assumptions of the models is given in Table 1.

Later work showed that most of the assumptions in these early models could be relaxed: Lintner (1969) incorporated heterogeneous beliefs. Brennan (1970) incorporated the effects of taxation. Mayers (1972) allowed for concentrated portfolios through trading restrictions on risky assets, transactions costs, and information asymmetries. Black (1972a) utilized the two-funds separation theorem[23] to construct the zero-beta CAPM, by using a portfolio that is orthogonal to the market portfolio in place of a risk-free asset. Rubinstein (1973) extended the model to higher moments, and also derived the CAPM without a riskless asset. Ingersoll (1975) and Kraus and Litzenberger (1976) also incorporated the higher moments.

The models of Treynor (1962), Sharpe (1964), Lintner (1965a), and Mossin (1966) have much in common. Table 2 summarizes the models'

Table 2 Characteristics of the models.

Model type	Single-period/multi-period	Discrete time/continuous time	Market/consumption oriented	Mean–variance objective function
Treynor (1962)	Single	Discrete	Market	Yes[a]
Sharpe (1964)	Single	Discrete	Market	Yes[b]
Lintner (1965)	Single	Discrete	Market	Yes[b]
Mossin (1966)	Single	Discrete	Market	Yes[a]

Requirements	Requires market clearing	Requires nonsingular covariance matrix	Allows short sales	Allows leverage
Treynor (1962)	Implicit	Implicit	Yes	Yes
Sharpe (1964)	No	No	No	No
Lintner (1965)	Implicit	Yes	Yes	Yes
Mossin (1966)	Explicit	Yes	Yes	Not addressed

Conclusions	Market itself is efficient	In equilibrium, the same combination of risky assets will be optimal for every investor	Amount invested in each risky asset will equal the ratio of market value of the asset to the total market value of all assets	Amount invested in each risky asset will be a positive amount
Treynor (1962)	Yes	Yes	Yes	Yes
Sharpe (1964)	No	No	No	Yes
Lintner (1965)	Yes	Yes	Yes	Yes
Mossin (1966)	Yes	Yes	Yes	Yes

Exposition method	Employs first-order conditions
Treynor (1962)	Yes
Sharpe (1964)	No
Lintner (1965)	Yes
Mossin (1966)	Yes

[a] Objective function stated in terms of terminal wealth and variance.
[b] Objective function stated in terms of percent return and standard deviation.

characteristics. All are single-period, discrete-time models,[24] and all are market-focused as opposed to consumption-focused. The most fundamental similarities are that each rest on the foundations of Markowitz (1952, 1959) and Tobin (1958), which both build upon the utility-of-wealth literature that assumes that agents are risk-averters with convex[25] loci of constant expected utility of wealth, represented as indifference curves in the mean–variance plane.

Utility-of-wealth notions are based, primarily, on the works of Friedman and Savage (1948), Marschak (1950), von Neumann and Morgenstern

(1953), and Savage (1954). Sharpe (1964) notes that Hirshleifer (1963) suggests that this model of investor behavior should be regarded as a special case of the more general constructs of Arrow (1953).[26] While Markowitz (1952) did not address the issue of probability beliefs, Markowitz (1959) did.[27] The critical departure of Mr. Treynor (and later, of Professors Sharpe, Lintner, and Mossin) from Dr. Tobin, aside from the purpose of the paper,[28] was the invocation of two key additional assumptions: The existence of a perfect lending market, and homogeneous expectations.[29]

It was the utilization of these two extremely restrictive and unrealistic assumptions that allowed Mr. Treynor to answer the compelling, yet unasked and unanswered, question in Tobin (1958): "*In equilibrium, what is the composition of E?*"[30] Tobin described investors as considering the universe of risky assets "as if there were a single non-cash asset, a composite formed by combining the multitude of actual non-cash assets in fixed proportions."[31] One of Mr. Treynor's primary accomplishments in developing the CAPM was to ask the question, "what are those 'fixed proportions'?", and answer: "… ideally, the investor will hold shares in each equity in proportion to the total number of shares available in the market."[32] Both Dr. Lintner and Dr. Mossin also reached such a conclusion.[33]

Dr. Fama's 1968 discussion of the Sharpe model agrees exactly with Dr. Sharpe's own discussion only up to the bottom of Fama (1968) p. 32, at which point Dr. Fama begins to deviate from Dr. Sharpe's discussion and to offer his own clarifying interpretation:

> … optimum portfolios for all investors will involve some combination of the riskless asset F and the portfolio of risky assets M. There will be no incentive to hold risky assets not in M. If M does not contain all the risky assets in the market, or if it does not contain them in exactly the proportions in which they are outstanding, then there will be some assets that no one will hold. This is inconsistent with equilibrium, since in equilibrium all assets must be held. Thus … M must be the market portfolio; that is, M consists of all risky assets in the market, each weighted by the ratio of its total market value to the total market value of all risky assets … The market portfolio M is the only efficient portfolio of risky assets.[34]

This is a valid conclusion if we require that markets clear, but it is one that Dr. Sharpe himself did not make, whereas Mr. Treynor, Dr. Lintner, and Dr. Mossin did.

While Tobin (1958) restricts holdings of "consols," or risky assets, to long holdings only (p. 82, "all x_i are non-negative," where x_i represents the weight in the portfolio of the ith non-cash asset), Treynor (1962) does not (p. 16 explicitly allows for short selling, though in his equilibrium result there would be none, as noted on p. 21). Sharpe (1964), in contrast, explicitly disallows short sales in the model.[35] We interpret Dr. Sharpe's observation on p. 437 that "a combination [of asset i plus an efficient combination of assets g] in which asset i does not appear at all must be represented by some negative value of α" not expressly as allowing the overt negative holding of asset i, but rather as a device which allows us to interpret point g' in such a fashion, without any *actual* short sale having occurred—that is, since g' is the portfolio g excluding any holding in asset i, we can consider g' to be the combination $[-]\alpha i + (1 - [-]\alpha)g$. Both Lintner (1965a, p. 19) and Mossin (1966, p. 776) allow for short sales.

Tobin (1958) does not cover the case of borrowing—he explicitly disallows leverage (p. 82, $\sum x_i = 1$);[36] the portfolio is restricted to lending only, extending the "opportunity locus" only to point E (p. 83, Figure 3.6) where $\sum x_i = 1$, whereas Treynor (1962) extends the "efficient set" beyond $\sum x_i = 1$ (p. 16, "Another aspect of the present paper which diverges from the Tobin paper is the absence of positivity constraints. The individual investor is free to borrow or lend, to buy long—or sell short—as he chooses …").

While Sharpe (1964) disallows leverage via his nonnegativity constraint on all assets (including the risk-free asset), he discusses the possibility (p. 433, "If the investor can borrow ... this is equivalent to disinvesting in [the risk-free asset]. The effect of borrowing ... can be found simply by letting α [the proportion of wealth invested in P, the riskless asset] take on negative values ..."). Lintner (1965a) also allows for borrowing (p. 15), while Mossin (1966) is silent on this issue.

All of the theorists express optimal portfolios using the vector of (expected mean) returns and the covariance/variance matrix; as Dr. Markowitz examines risky assets only, his E, V efficient set is therefore nonlinear, whereas Dr. Tobin, Mr. Treynor, Dr. Sharpe, Dr. Lintner, and Dr. Mossin employ a riskless asset in order to derive a linear opportunity locus/efficient set/capital market line/market opportunity line/market line. In order to derive the opportunity locus, or "ray of dominant sets," Tobin (1958) makes use of Lagrange multipliers in order to minimize variance per expected mean return (p. 83). Dr. Markowitz' critical line algorithm makes use of the same technique in the computation of efficient sets (1959, Appendix A). Treynor (1962), Lintner (1965a), and Mossin (1966) all use the same technique.

4 Conclusion

Bernstein (1992) quotes Franco Modigliani, referring to his mentoring of Mr. Treynor, as saying "I made a mistake with Treynor. He was trying to bite off so big a bullet that I did not give sufficient stress to the one part that was right." That "one part" was Treynor (1962). Dr. Modigliani was not the only one to initially miss the power and elegance of the CAPM; when Dr. Sharpe first submitted his manuscript for Sharpe (1964) to the *Journal of Finance* in 1962, a referee recommended to the editor that, due to its extremely restrictive assumptions (primarily Dr. Sharpe's second equilibrium assumption—that all investors hold the same views regarding future expected values, standard deviations, and correlation coefficients—a restriction that was subsequently dubbed "homogeneity of investor expectations" by one of the referees[37]), the paper not be published, as it was "uninteresting."[38]

Current researchers almost never cite Treynor (1962). Research citation tends to evolve in Darwinian fashion. Later researchers have little incentive to reference a paper that is not cited by earlier ones. The early important works of Dr. Lintner, Dr. Mossin, Dr. Fama, and Dr. Merton all ignore Mr. Treynor's early contributions, and later theoreticians who refer to and extend these works are unlikely to cite Treynor (1962), simply because of path dependence.

The existing copies of Treynor (1962) in its original "Rough Draft" form are not publicly available, though fortunately copies do exist in private collections. It appears that, because it was unpublished until recently, Mr. Treynor's early work has not been widely distributed, and is therefore cited less frequently. Perhaps the publication of Treynor (1962) as chapter 2 of Korajczyk (1999) will mitigate this issue. It seems that, as Fischer Black stated, Mr. Treynor developed the first CAPM, and that he should be more widely credited with its invention.

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Notes

1. Black (1981), p. 14.
2. See Sharpe and Alexander (1978), p. 194; Merton (1990), p. 475; Bodie *et al.* (1993), p. 242; Reilly (1994), p. 270; and Cochrane (2001), p. 152.
3. Please refer to the final version of Treynor (1962), which was published as chapter 2 of Korajczyk (1999). All page and paragraph references herein regarding Treynor (1962) specify those in the Korajczyk (1999) version. Treynor (1961) remains unpublished.
4. Although Mr. Treynor's 1962 paper, "Toward a Theory of Market Value of Risky Assets" is correctly referred to as "Treynor (1962)" in Korajczyk (1999); it is also occasionally referred to as "Treynor (1963)", e.g., Harrington and Korajczyk (1993), p. 123. Some references append an "s" to the "Toward", e.g., Luenberger (1998), while most do not; however, the published version in Korajczyk (1999) includes the "s" in its title. The author's copy of the 1962 mimeograph does not. We have yet to find a reference to "Market Value, Time, and Risk" in the literature.
5. In experiment space, there is one dimension, or axis, for each experiment. For example, the correlation coefficient of two time series is the cosine of the angle between two vectors, and statistically independent random variables are orthogonal in experiment space.
6. This paragraph relies on personal correspondence with Jack Treynor, and parallels the discussion in Bernstein (1992), pp. 183–202. Note that Bernstein (1992) reports the year of Mr. Treynor's vacation as 1959, but the correct year is 1958.
7. Refer to Korajczyk (1999), p. 20.
8. Jensen (1972b), p. 4.
9. Black *et al.* (1972), p. 79.
10. Black (1972b), p. 249.
11. Ross (1987), p. 24.
12. Mr. Treynor was not alone in employing this convention; Dr. Lintner also used this notation—see Lintner (1965a), p. 20, Eqs. (6b) and (8). Please note that all notation in the present paper is as defined in the original models; we do not introduce any new notation, nor do we redefine any of the original conventions. The reader's understanding will be enhanced by direct reference to the original papers in this discussion.
13. Page and paragraph locations refer to the published version of Treynor (1962) in Korajczyk (1999).
14. Personal communication of Mr. Treynor to the author.
15. Modigliani and Miller (1958), p. 268, formula (3): $V_j \equiv (S_j + D_j) = X_j/\rho_k$.
16. Another interesting aspect of Mossin (1966) is that Mossin's discussion of structural diversification indifference on pages 779–781 is the first formal proof of the "homemade diversification" argument. This was noted in Rubinstein (1973).
17. Lintner notes, "*Positive* (negative) *risk premiums are neither a sufficient nor a necessary condition for a stock to be held long* (short)" [original emphasis.] Lintner (1965a, p. 23).
18. Markowitz (1952), p. 80: $\sigma_{ij} = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$.
19. Markowitz (1952), p. 81: $V(R) = \sum\sum \alpha_i \alpha_j \sigma_{ij}$.
20. The Lagrange multiplier λ is the marginal utility of wealth, given constant prices.
21. Tobin (1958), p. 84: $\mu_R = \sigma_R \left(\sum_i \sum_j r_i r_j V_{ij}\right)^{1/2}$.
22. Lintner (1965a), p. 25.
23. Markowitz (2000) distinguishes between the *Tobin separation theorem* and the *two-funds separation theorem*: First, Tobin (1958) showed that the choice of proportions among risky assets is a separate decision from the choice of leverage—every efficient portfolio is of the form $x = \alpha x^* + (1-\alpha)x^c$, where x^c is the risk-free asset, x^* is a

portfolio of risky assets and α is a non-negative scalar. This result is known as the Tobin separation theorem. Later, Sharpe (1970) and Merton (1972) showed that if the only constraint is $\sum x_i = 1$, then every efficient portfolio is the combination $x = \alpha x^* + (1-\alpha)x$ with $\alpha \geq 0$, where x is the minimum-variance efficient portfolio, with or without the existence of a risk-free asset. This result is the two-funds separation theorem [Markowitz (2000), pp. 38–39]. Merton (1990) provides the generalized "*three-fund*" *separation theorem*, which asserts indifference among agents between portfolios selected from the original n assets or portfolios composed of (1) the market portfolio, (2) the riskless asset, and (3) a portfolio that is instantaneously perfectly correlated with changes in the interest rate [Merton (1990), pp. 382–386 and 490–492.]

24 Treynor (1961, 1963) also address multi-period and continuous-time environments.

25 Although Tobin terms this "concave upwards", we refer here to the currently accepted definition of concavity/convexity of functions in Euclidean space.

26 Sharpe (1964), p. 427, footnote 6.

27 See, in particular, Markowitz (1952), p. 81, footnote 7: "This paper does not consider the difficult question of how investors do (or should) form their probability beliefs," and Markowitz (1959), chapter 10, which outlines three axioms of rational behavior, as well as chapter 13, which addresses what Markowitz calls the second "chief limitation" of Markowitz (1952), the assumption of static probability beliefs.

28 Tobin sought to explain the demand for cash and its inverse relationship with the differential in the yields of default-free fixed income instruments, thus maintaining the implications of the liquidity preference component of Keynes' theory of underemployment equilibrium. The "risk aversion theory of liquidity preference" in Tobin (1958) was developed to avoid the objectionable properties of Keynes' (1936) assumption of stickiness in interest rate expectations, which had been criticized in Fellner (1946) and Leontief (1947). The risk in Tobin's "consols" is uncertainty about future interest rates only. In contrast, Mr. Treynor and subsequent financial economists sought to develop a general equilibrium model of capital asset price behavior, and were primarily concerned with the risk of equity price fluctuations; in doing so, they implicitly assumed away the only risk Tobin considered (see Treynor (1962), p. 16).

29 See Treynor (1962), p. 15, assumptions 6 and 7; also see Sharpe (1964), p. 433, two assumptions for equilibrium. Mr. Treynor's terms for these assumptions are a "perfect lending market" and "perfect [investor] knowledge"; Sharpe's terms are "a common pure rate of interest" and "homogeneity of investor expectations."

30 Here E is the point where $\sum x_i = 1$ on the ray of dominant sets, illustrated in Tobin (1958), p. 83, Figure 3.6.

31 Tobin (1958), p. 84. This composite is represented graphically in Tobin by E (p. 83, Figure 3.6), in Sharpe by ϕ (p. 432, Figure 4), and in Lintner by M (p. 19, Figure 1); it is not graphically illustrated either in Treynor or in Mossin.

32 Treynor (1962), p. 21.

33 Lintner (1965a), p. 25, and Mossin (1966), pp. 775–776.

34 Fama (1968), pp. 32–33. Fama emphasizes, in footnote 11 on p. 33, that Sharpe's version of equilibrium does not imply that the market portfolio M is the only efficient portfolio of risky assets.

35 Sharpe (1964), p. 433, footnote 15: "The discussion in this paper is based on Markowitz' formulation, which includes non-negativity constraints on the holdings of all assets." Markowitz' assumptions of no short sales ($X_j \geq 0$ for all j) and no leverage ($\sum X_j = 1$) can be found on p. 171 of Markowitz (1959).

36 However, Sharpe (1964) interprets otherwise; see p. 433, footnote 15 for Sharpe's relaxation of the no-leverage restriction of Tobin (1958).

37 Sharpe (1964), p. 433, footnote 16.

38 Bernstein (1992), pp. 194–195.

References

Arrow, K. J. (1953). "Le Rôle des Valuers Boursières pour la Répartition la Meilleure des Risques." *Économetrie*, Colloques Internationaux du Centre National de la Recherche Scientifique, vol. 11, Paris, pp. 41–47.

Bernstein, P. L. (1992). *Capital Ideas: The Improbable Origins of Modern Wall Street*. New York: The Free Press.

Black, F. (1972a). "Capital Market Equilibrium with Restricted Borrowing." *The Journal of Business* 45(3), 444–455.

Black, F. (1972b). "Equilibrium in the Creation of Investment Goods Under Uncertainty." In *Studies in the Theory of Capital Markets*. Michael C. Jensen (editor) New York: Praeger, pp. 249–265.

Black, F. (1981). "An Open Letter to Jack Treynor." *The Financial Analysts Journal* July/August. Letters to the Editor, p. 14.

Black, F., Jensen, M. C. and Scholes, M. (1972). "The Capital Asset Pricing Model: Some Empirical Tests." In *Studies in the Theory of Capital Markets*. Michael C. Jensen (editor) New York: Praeger, pp. 79–121.

Black, F. and Scholes, M. (1973). "The Pricing of Options and Corporate Liabilities." *The Journal of Political Economy* 81(3), 637–659.

Bodie, Z., Kane, A. and Marcus, A. J. (1993). *Investments*. 2nd edn. Burr Ridge: Irwin. (special edition for CFA candidates).

Brealey, R. A. and Edwards, H. (1991). *A Bibliography of Finance*. Cambridge: MIT Press.

Brennan, M. J. (1970). "Taxes, Market Valuation, and Corporate Financial Policy." *National Tax Journal* 4, 417–427.

Cochrane, J. H. (2001). *Asset Pricing*, Princeton: Princeton University Press.

Fama, E. F. (1968). "Risk, Return, and Equilibrium: Some Clarifying Comments." *The Journal of Finance* 23(1), 29–40.

Fama, E. F. and MacBeth, J. (1973). "Risk, Return and Equilibrium: Empirical Tests." *The Journal of Political Economy* 81(3), 607–636.

Fellner, W. J. (1946). *Monetary Policies and Full Employment*. Berkeley: University of California Press.

Fisher, I. (1930). *The Theory of Interest*. New York: MacMillan.

Friedman, M. and Savage, L. J. (1948). "The Utility Analysis of Choices Involving Risk." *The Journal of Political Economy* 56(4), 279–304.

Harrington, D. R. and Korajczyk, R. A. (editors) (1993). *The CAPM Controversy: Policy and Strategy Implications for Investment Management*. Charlottesville: Association for Investment Management and Research; ICFA Continuing Education.

Hirshleifer, J. (1963). "Investment Decision Under Uncertainty", *Papers and Proceedings of the Seventy-Sixth Annual Meeting of the American Economic Association*, December. Subsequently extended and published as "Investment Decision Under Uncertainty: Choice-Theoretic Approaches." *The Quarterly Journal of Economics* 79(5), 509–536. Also, see "Investment Decision Under Uncertainty: Applications of the State-Preference Approach." *The Quarterly Journal of Economics* 80(2), 252–277.

Ingersoll, J. E., Jr. (1975). "Multidimensional Security Pricing." *The Journal of Financial and Quantitative Analysis* 10, 785–798.

Jensen, M. C. (1968). "The Performance of Mutual Funds in the Period 1945–64." *The Journal of Finance* 23(2), 389–416.

Jensen, M. C. (1969). "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios." *The Journal of Business* 42(2), 167–247.

Jensen, M. C. (editor) (1972a). *Studies in the Theory of Capital Markets*. New York: Praeger.

Jensen, M. C. (1972b). "The Foundations and Current State of Capital Market Theory." In *Studies in the Theory of Capital Markets*. Michael C. Jensen (editor) New York: Praeger, pp. 3–43.

Keynes, J. M. (1936). *The General Theory of Employment, Interest, and Money*. New York: Harcourt Brace.

Korajczyk, R. A. (1999). *Asset Pricing and Portfolio Performance: Models, Strategy and Performance Metrics*. London: Risk Books.

Kraus, A. and Litzenberger, R. H. (1976). "Skewness Preference and the Valuation of Risk Assets." *The Journal of Finance* 31, 1085–1100.

Leontief, W. (1947). "Postulates: Keynes' General Theory and the Classicists." In *The New Economics: Keynes' Influence on Theory and Public Policy*, chapter 19. Seymour E. Harris (editor) New York: Knopf, pp. 232–242.

Lintner, J. (1965a). "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *The Review of Economics and Statistics* 47, 13–37.

Lintner, J. (1965b). "Securities Prices, Risk, and Maximal Gains from Diversification." *The Journal of Finance* 20(4), 587–615.

Lintner, J. (1969). "The Aggregation of Investor's Diverse Judgement and Preferences in Purely Competitive Securities Markets." *The Journal of Financial and Quantitative Analysis* 4, 347–400.

Lintner, J. (1972). *Finance and Capital Markets*. New York: National Bureau of Economic Research.

Long, J. B. Jr. (1972). "Consumption-Investment Decisions and Equilibrium in the Securities Markets." In *Studies in the Theory of Capital Markets*. Michael C. Jensen (editor) New York: Praeger, pp. 146–222.

Luenberger, D. G. (1998). *Investment Science*. Oxford: Oxford University Press.

Markowitz, H. M. (1952). "Portfolio Selection." *The Journal of Finance* 7(1), 77–91.

Markowitz, H. M. (1959). *Portfolio Selection: Efficient Diversification of Investments*, Cowles Foundation for Research in Economics at Yale University, Monograph #6. New York: John Wiley & Sons, Inc. (Second edition, 1991, Cambridge: Basil Blackwell, Inc.).

Markowitz, H. M. (2000). *Mean–Variance Analysis in Portfolio Choice and Capital Markets*. New Hope: Frank J. Fabozzi Associates.

Marschak, J. (1950). "Rational Behavior, Uncertain Prospects, and Measurable Utility." *Econometrica* 18(2), 111–141.

Mayers, D. (1972). "Nonmarketable Assets and Capital Market Equilibrium Under Uncertainty." In *Studies in the Theory* *of Capital Markets*. Michael C. Jensen (editor) New York: Praeger, pp. 223–248.

Merton, R. C. (1972). "An Analytic Derivation of the Efficient Portfolio Frontier." *The Journal of Financial and Quantitative Analysis* 7, 1851–1872.

Merton, R. C. (1990). *Continuous-Time Finance*. Cambridge: Blackwell (revised paperback edition, 1999 reprint).

Modigliani, F. and Miller, M. H. (1958). "The Cost of Capital, Corporation Finance, and the Theory of Investment." *The American Economic Review* 48, 261–297.

Mossin, J. (1966). "Equilibrium in a Capital Asset Market." *Econometrica* 34(4), 768–783.

Mossin, J. (1973). *Theory of Financial Markets*. Englewood Cliffs: Prentice-Hall.

Mossin, J. (1977). *The Economic Efficiency of Financial Markets*. Lanham: Lexington.

Reilly, F. K. (1994). *Investment Analysis and Portfolio Management*, 4th edn. Fort Worth: Dryden.

Roll, R. (1977). "A Critique of the Asset Pricing Theory's Tests Part I: On Past and Potential Testability of the Theory." *The Journal of Financial Economics* 4(2), 129-176.

Ross, S. A. (1976). "The Arbitrage Theory of Capital Asset Pricing." *The Journal of Economic Theory* 13(3), 341–360.

Ross, S. A. (1977). "Return, Risk and Arbitrage." In *Risk and Return in Finance*, Section 9. Irwin Friend and J. Bicksler (editors) Cambridge: Ballinger, pp. 189–219.

Ross, S. A. (1987). "Finance." In *The New Palgrave Finance*. John Eatwell, Murray Milgate and Peter Newman (editors) New York: W.W. Norton, pp. 1–34.

Rubinstein, M. (1973). "The Fundamental Theorem of Parameter-Preference Security Valuation." *The Journal of Financial and Quantitative Analysis* 8, 61–69.

Savage, L. J. (1954). *The Foundations of Statistics*. New York: John Wiley & Sons.

Sharpe, W. F. (1961). "Portfolio Analysis Based on a Simplified Model of the Relationships Among Securities." Unpublished doctoral dissertation. Los Angeles: University of California at Los Angeles.

Sharpe, W. F. (1963). "A Simplified Model for Portfolio Analysis." *Management Science* 9(2), 277–293.

Sharpe, W. F. (1964). "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk." *The Journal of Finance* 19(3), 425–442.

Sharpe, W. F. (1966). "Mutual Fund Performance." *The Journal of Business* 39, 119–138.

Sharpe, W. F. (1970). *Portfolio Theory and Capital Markets*. New York: McGraw-Hill.

Sharpe, W. F. (1990). "Autobiography." In *Les Prix Nobel 1990*. Tore Frängsmyr (editor) Stockholm: Nobel Foundation.

Sharpe, W. F. and Alexander, G. J. (1978). *Investments*, 4th edn. (1990). Englewood Cliffs: Prentice-Hall.

Stone, B. K. (1970). *Risk, Return, and Equilibrium; A General Single-Period Theory of Asset Selection and Capital-Market Equilibrium*. Cambridge: MIT Press.

Tobin, J. (1958). "Liquidity Preference as Behavior Towards Risk." *The Review of Economic Studies*, No. 67, February, 65–86. Reprinted as Cowles Foundation Paper 118.

Treynor, J. L. (1961). "Market Value, Time, and Risk." Unpublished manuscript. "Rough Draft" dated 8/8/61, #95–209.

Treynor, J. L. (1962). "Toward a Theory of Market Value of Risky Assets." Unpublished manuscript. "Rough Draft" dated by Mr. Treynor to the fall of 1962. A final version was published in 1999, in *Asset Pricing and Portfolio Performance*. Robert A. Korajczyk (editor) London: Risk Books, pp. 15–22.

Treynor, J. L. (1963). "Implications for the Theory of Finance." Unpublished manuscript. "Rough Draft" dated by Mr. Treynor to the spring of 1963.

Treynor, J. L. (1965). "How to Rate Management of Investment Funds." *Harvard Business Review* 43, 63–75.

Treynor, J. L. and Mazuy, K. (1966). "Can Mutual Funds Outguess the Market?" *Harvard Business Review* 44, 131–136.

Treynor, J. L. and Black, F. (1973). "How to Use Security Analysis to Improve Portfolio Selection." *The Journal of Business* 46(1), 66–88.

Treynor, J. L. and Black, F. (1976). "Corporate Investment Decisions." In *Modern Developments in Financial Management*. Stewart C. Myers (editor) Hinsdale: The Dryden Press, pp. 310–327.

von Neumann, J. L. and Morgenstern, O. (1953). *Theory of Games and Economic Behavior*, 3rd edn. Princeton: Princeton University Press.